

# QUIVER VARIETIES AND TENSOR PRODUCTS

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**ABSTRACT.** In this article, we give geometric constructions of tensor products in various categories using quiver varieties. More precisely, we introduce a lagrangian subvariety  $\tilde{\mathfrak{Z}}$  in a quiver variety, and show the following results: (1) The homology group of  $\tilde{\mathfrak{Z}}$  is a representation of a symmetric Kac-Moody Lie algebra  $\mathfrak{g}$ , isomorphic to the tensor product  $V(\lambda_1) \otimes \cdots \otimes V(\lambda_N)$  of integrable highest weight modules. (2) The set of irreducible components of  $\tilde{\mathfrak{Z}}$  has a structure of a crystal, isomorphic to that of the  $q$ -analogue of  $V(\lambda_1) \otimes \cdots \otimes V(\lambda_N)$ . (3) The equivariant  $K$ -homology group of  $\tilde{\mathfrak{Z}}$  is isomorphic to the tensor product of universal standard modules of the quantum loop algebra  $\mathbf{U}_q(\mathbf{Lg})$ , when  $\mathfrak{g}$  is of type *ADE*. We also give a purely combinatorial description of the crystal of (2). This result is new even when  $N = 1$ .

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## INTRODUCTION

Let  $\mathfrak{g}$  be a symmetric Kac-Moody Lie algebra and let  $\mathbf{U}_q(\mathfrak{g})$  be the quantum enveloping algebra of Drinfeld-Jimbo attached to  $\mathfrak{g}$ . For each dominant weight  $\mathbf{w}$  of  $\mathfrak{g}$ , the author associated a nonsingular variety  $\mathfrak{M}(\mathbf{w})$  (called a *quiver variety*), containing a half dimensional subvariety  $\mathfrak{L}(\mathbf{w})$  [28, 30]. It is related to the representation theory of  $\mathfrak{g}$  and  $\mathbf{U}_q(\mathfrak{g})$  as follows:

- (1) The top degree homology group  $H_{\text{top}}(\mathfrak{L}(\mathbf{w}), \mathbb{C})$  has a structure of a  $\mathfrak{g}$ -module, simple with highest weight  $\mathbf{w}$ . ([30])
- (2) The set  $\text{Irr } \mathfrak{L}(\mathbf{w})$  of irreducible components of  $\mathfrak{L}(\mathbf{w})$  has a structure of a crystal, isomorphic to the crystal of the simple  $\mathbf{U}_q(\mathfrak{g})$ -module with highest weight  $\mathbf{w}$ . (Kashiwara-Saito [16, 35])

Suppose  $\mathfrak{g}$  is of type *ADE*. Let  $\mathbf{Lg} = \mathfrak{g}[x, x^{-1}]$  be the loop algebra of  $\mathfrak{g}$  and let  $\mathbf{U}_q(\mathbf{Lg})$  be the quantum loop algebra (the quantum affine algebra without central extension and degree

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operator) with the integral form  $\mathbf{U}_q^{\mathbb{Z}}(\mathbf{L}\mathbf{g})$  generated by  $q$ -divided powers. In [32], the author showed the following:

- (3) The equivariant  $K$ -homology group  $K^{H_{\mathbf{w}} \times \mathbb{C}^*}(\mathfrak{L}(\mathbf{w}))$  has a structure of  $\mathbf{U}_q^{\mathbb{Z}}(\mathbf{L}\mathbf{g})[x_1^{\pm}, \dots, x_L^{\pm}]$ -module, which is simple and has various nice properties.

Here  $L = \sum_k \langle h_k, \mathbf{w} \rangle$  and  $H_{\mathbf{w}}$  is a torus of dimension  $L$  acting on  $\mathfrak{M}(\mathbf{w})$  and  $\mathfrak{L}(\mathbf{w})$ . We call it a *universal standard module*, and denote it by  $M(\mathbf{w})$ . (In the main body of this article, we replace  $H_{\mathbf{w}}$  by a product of general linear groups, and  $\mathbf{U}_q^{\mathbb{Z}}(\mathbf{L}\mathbf{g})[x_1^{\pm}, \dots, x_L^{\pm}]$  by the invariant part of a product of symmetric groups.) It was shown that any simple  $\mathbf{U}_q(\mathbf{L}\mathbf{g})$ -module is obtained as a quotient of a specialization of  $M(\mathbf{w})$ . This specialization is called a *standard module*. Moreover, the multiplicities of simple modules in standard modules are computable by a combinatorial algorithm [34].

In this article, we generalize results (1),(2),(3) to the case of tensor products of simple modules. For given dominant weights  $\mathbf{w}^1, \mathbf{w}^2, \dots, \mathbf{w}^N$  we introduce another half-dimensional subvariety  $\tilde{\mathfrak{Z}}$  of  $\mathfrak{M}(\mathbf{w})$ , containing  $\mathfrak{L}(\mathbf{w})$  ( $\mathbf{w} = \mathbf{w}^1 + \dots + \mathbf{w}^N$ ), and show the followings:

- (1)'  $H_{\text{top}}(\tilde{\mathfrak{Z}}, \mathbb{C})$  has a structure of a  $\mathfrak{g}$ -module, isomorphic to the tensor product of simple  $\mathfrak{g}$ -modules with highest weights  $\mathbf{w}^1, \dots, \mathbf{w}^N$ . (Theorem 5.2)
- (2)' The set  $\text{Irr } \tilde{\mathfrak{Z}}$  of irreducible components of  $\tilde{\mathfrak{Z}}$  has a structure of a crystal, isomorphic to the tensor product of the crystals  $\text{Irr } \mathfrak{L}(\mathbf{w}^1), \dots, \text{Irr } \mathfrak{L}(\mathbf{w}^N)$ . (Theorem 4.6)
- (3)' When  $\mathfrak{g}$  is of type  $ADE$ , the equivariant  $K$ -homology group  $K^{H_{\mathbf{w}} \times \mathbb{C}^*}(\tilde{\mathfrak{Z}})$  is isomorphic to the tensor product of universal standard modules  $M(\mathbf{w}^1), \dots, M(\mathbf{w}^N)$ . (Theorem 6.11)

The result (2)' means that  $\{\text{Irr } \mathfrak{L}(\mathbf{w}) \mid \mathbf{w} \in P^+\}$  is a closed family of highest weight normal crystals (see §1.2 for definition). This property characterizes crystals of simple highest weight modules. Thus we obtain a new proof of (2).

As an application of (3)', we give a new proof of the main result of Varagnolo-Vasserot [37] (see Corollary 6.12). It says that a standard module is a tensor product of fundamental representations in an appropriately chosen order.

We also give a combinatorial description of the crystal  $\text{Irr } \mathfrak{L}(\mathbf{w})$  in §8. It is essentially the same as a combinatorial description of  $V(\mathbf{w})$ , provided by the embedding theorem of the crystal  $\mathcal{B}(\infty)$  of the lower part of the quantized enveloping algebra ([14]). We shall discuss this description further, relating it with the theory of  $q$ -characters in a separate publication.

The varieties  $\tilde{\mathfrak{Z}}, \mathfrak{Z}$  are defined as attracting sets of a  $\mathbb{C}^*$ -action for some one parameter subgroup  $\lambda: \mathbb{C}^* \rightarrow H_{\mathbf{w}}$ . The relation between the  $\mathbb{C}^*$ -action and tensor products has been known to many people after the author related the quiver varieties to Lusztig's construction of canonical bases [28], since Lusztig defined the comultiplication in a geometric way (see [20]). Moreover, when  $\mathfrak{g}$  is of type  $A_n$  and  $\mathbf{w}$  is a multiple of the fundamental weight corresponding to the vector representation, the quiver variety  $\mathfrak{M}(\mathbf{w})$  is isomorphic to the cotangent bundle of the  $n$ -step partial flag variety. In this case the comultiplication was constructed by Ginzburg-Reshetikhin-Vasserot [7]. The result was also mentioned without detail in an earlier paper by Grojnowski [8]. And the author used the  $\mathbb{C}^*$ -action to compute Betti numbers of  $\mathfrak{M}(\mathbf{w})$  (when  $\mathfrak{g}$  is of type  $A$ ), and checked that the generating function of the Euler numbers is the character of the tensor product [29]. For general  $\mathfrak{g}$ , Grojnowski mentioned, in his 'advertisement' [9] of his book, that the coproduct is defined by the localization to the fixed point set of the  $\mathbb{C}^*$ -action. However the details of the construction of  $\mathbf{U}_q(\mathbf{L}\mathbf{g})$ -module structures were not given. The details were given in [32], and by the localization, tensor products were studied, but only for generic parameters [14.1.2, loc. cit.] (see also Lemma 6.4). Tensor products for arbitrary

parameters need further study, and it was first done by Varagnolo-Vasserot [37]. The result (3)' above is motivated by their study, although the author knew  $\tilde{\mathfrak{Z}}$  before their paper appeared.

The variety  $\tilde{\mathfrak{Z}}$  is an analogue of subvarieties of cotangent bundles of flag varieties, introduced by Lusztig [23]. (Our notation is taken from his.) In his picture,  $\mathfrak{M}(\mathbf{w})$  corresponds to Slodowy's variety,  $\mathfrak{L}(\mathbf{w})$  to the Springer fiber  $\mathcal{B}_x$ . The equivariant  $K$ -homology group of  $\mathcal{B}_x$  is a module of the affine Hecke algebra  $\widehat{H}_q$ . The equivariant  $K$ -homology of  $\tilde{\mathfrak{Z}}$  is an induced module of a module of above type for a smaller affine Hecke algebra.

One of motivations of [23] was a conjectural construction of a base in the equivariant  $K$ -homology of  $\mathcal{B}_x$ . Lusztig pointed out a possibility of a similar construction for quiver varieties [25]. Combining a result in [32] with a recent result by Kashiwara [15], the universal standard module  $M(\mathbf{w})$  has a global crystal base. It is interesting to compare his base with Lusztig's (conjectural) base.

Finally we comment that there is a geometric construction of tensor products of *two* simple  $\mathfrak{g}$ -modules by Lusztig [26]. His variety is a subvariety of a product of quiver varieties. In fact, an open subvariety of an analogue of Steinberg variety, which will be denoted by  $Z(\mathbf{w})$  in this article. The relation between his variety and  $\tilde{\mathfrak{Z}}$  is not clear, although there is an example where a close relation can be found (see §9). And it seems difficult to generalize his varieties to the case of tensor products of several modules. (Compare §7).

After this work was done, we were informed that Malkin also defined the variety  $\tilde{\mathfrak{Z}}$  and obtained the result (2)' above [27].

**Acknowledgement.** This work originated in G. Lusztig's question about analogue of his variety in quiver varieties, asked at the Institute for Advanced Study, 1998 winter. It is a great pleasure for me to answer his question after two years. I would like to thank I. Grojnowski for explaining me his construction of tensor products for generic parameters during 2000 winter. I also express my sincere gratitude to M. Kashiwara for interesting discussions about crystal.

## 1. PRELIMINARIES (I) – ALGEBRAIC PART

**1.1. Quantized enveloping algebra.** We briefly recall the definition of the quantized enveloping algebra in this subsection. See [13] for further detail.

A *root datum* consists of

- (1)  $P$  : free  $\mathbb{Z}$ -module (weight lattice),
- (2)  $P^* = \text{Hom}_{\mathbb{Z}}(P, \mathbb{Z})$  with a natural pairing  $\langle \cdot, \cdot \rangle : P \otimes P^* \rightarrow \mathbb{Z}$ ,
- (3) a finite set  $I$  (index set of simple roots)
- (4)  $\alpha_k \in P$  ( $k \in I$ ) (simple root),
- (5)  $h_k \in P^*$  ( $k \in I$ ) (simple coroot),
- (6) a symmetric bilinear form  $(\cdot, \cdot)$  on  $P$ .

Those are required to satisfy the followings:

- (a)  $\langle h_k, \lambda \rangle = 2(\alpha_k, \lambda)/(\alpha_k, \alpha_k)$  for  $k \in I$  and  $\lambda \in P$ ,
- (b)  $\mathbf{C} \stackrel{\text{def.}}{=} (\langle h_k, \alpha_l \rangle)_{k,l}$  is a symmetrizable generalized Cartan matrix, i.e.,  $\langle h_k, \alpha_k \rangle = 2$ , and  $\langle h_k, \alpha_l \rangle \in \mathbb{Z}_{\leq 0}$  and  $\langle h_k, \alpha_l \rangle = 0 \iff \langle h_l, \alpha_k \rangle = 0$  for  $k \neq l$ ,
- (c)  $(\alpha_k, \alpha_k) \in 2\mathbb{Z}_{>0}$ ,
- (d)  $\{\alpha_k\}_{k \in I}$  are linearly independent,
- (e) there exists  $\Lambda_k \in P$  ( $k \in I$ ) such that  $\langle h_l, \Lambda_k \rangle = \delta_{kl}$  (fundamental weight).

Let  $\mathfrak{g}$  be the symmetrizable Kac-Moody Lie algebra corresponding to the generalized Cartan matrix  $\mathbf{C}$  with the Cartan subalgebra  $\mathfrak{h} = P^* \otimes_{\mathbb{Z}} \mathbb{Q}$ . Let  $Q = \bigoplus_k \mathbb{Z}\alpha_k \subset P$  be the root

lattice. Let  $P^+$  be the semigroup of dominant weights, i.e.,  $P^+ = \{\lambda \mid \langle h_k, \lambda \rangle \geq 0\}$ . Let  $Q^+ = \sum_k \mathbb{Z}_{\geq 0} \alpha_k$ .

Let  $q$  be an indeterminate. For nonnegative integers  $n \geq r$ , define

$$[n]_q \stackrel{\text{def.}}{=} \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]_q! \stackrel{\text{def.}}{=} \begin{cases} [n]_q [n-1]_q \cdots [2]_q [1]_q & (n > 0), \\ 1 & (n = 0), \end{cases} \quad \begin{bmatrix} n \\ r \end{bmatrix}_q \stackrel{\text{def.}}{=} \frac{[n]_q!}{[r]_q! [n-r]_q!}.$$

The quantized universal enveloping algebra  $\mathbf{U}_q(\mathfrak{g})$  of the Kac-Moody algebra  $\mathfrak{g}$  is the  $\mathbb{Q}(q)$ -algebra generated by  $e_k, f_k$  ( $k \in I$ ),  $q^h$  ( $h \in P^*$ ) with relations

$$(1.1a) \quad q^0 = 1, \quad q^h q^{h'} = q^{h+h'},$$

$$(1.1b) \quad q^h e_k q^{-h} = q^{\langle h, \alpha_k \rangle} e_k, \quad q^h f_k q^{-h} = q^{-\langle h, \alpha_k \rangle} f_k,$$

$$(1.1c) \quad e_k f_l - f_l e_k = \delta_{kl} \frac{t_k - t_k^{-1}}{q_k - q_k^{-1}},$$

$$(1.1d) \quad \sum_{p=0}^b (-1)^p e_k^{(p)} e_l e_k^{(b-p)} = \sum_{p=0}^b (-1)^p f_k^{(p)} f_l f_k^{(b-p)} = 0 \quad \text{for } k \neq l,$$

where  $q_k = q^{(\alpha_k, \alpha_k)/2}$ ,  $t_k = q^{(\alpha_k, \alpha_k)h_k/2}$ ,  $b = 1 - \langle h_k, \alpha_l \rangle$ ,  $e_k^{(p)} = e_k^p / [p]_{q_k}!$ ,  $f_k^{(p)} = f_k^p / [p]_{q_k}!$ .

Let  $\mathbf{U}_q(\mathfrak{g})^+$  (resp.  $\mathbf{U}_q(\mathfrak{g})^-$ ) be the  $\mathbb{Q}(q)$ -subalgebra of  $\mathbf{U}_q(\mathfrak{g})$  generated by elements  $e_k$ 's (resp.  $f_k$ 's). Let  $\mathbf{U}_q(\mathfrak{g})^0$  be the  $\mathbb{Q}(q)$ -subalgebra generated by elements  $q^h$  ( $h \in P^*$ ). We have the triangular decomposition  $\mathbf{U}_q(\mathfrak{g}) \cong \mathbf{U}_q(\mathfrak{g})^+ \otimes \mathbf{U}_q(\mathfrak{g})^0 \otimes \mathbf{U}_q(\mathfrak{g})^-$ .

Let  $\mathbf{U}_q^{\mathbb{Z}}(\mathfrak{g})$  be the  $\mathbb{Z}[q, q^{-1}]$ -subalgebra of  $\mathbf{U}_q(\mathfrak{g})$  generated by elements  $e_k^{(n)}, f_k^{(n)}, q^h$  for  $k \in I$ ,  $n \in \mathbb{Z}_{>0}$ ,  $h \in P^*$ .

In this article, we take the comultiplication  $\Delta$  on  $\mathbf{U}_q(\mathfrak{g})$  given by

$$(1.2) \quad \begin{aligned} \Delta q^h &= q^h \otimes q^h, \quad \Delta e_k = e_k \otimes q^{-h_k} + 1 \otimes e_k, \\ \Delta f_k &= f_k \otimes 1 + q^{h_k} \otimes f_k. \end{aligned}$$

Note that this is different from one in [20], although there is a simple relation between them [13, 1.4]. The results in [32] hold for either comultiplication (tensor products appear in (1.2.19) and (14.1.2)). In [34, §2] another comultiplication was used. If we reverse the order of the tensor product, the results hold.

For each dominant weight  $\lambda \in P^+$ , there is unique simple module  $V(\lambda)$  with highest weight  $\lambda$ . The highest weight vector is denoted by  $b_\lambda$ .

Later we use the classical counter part of  $\mathbf{U}_q(\mathfrak{g})$ . We just erase  $q$  in the above notation, e.g.,  $\mathbf{U}_q(\mathfrak{g}) \Rightarrow \mathbf{U}(\mathfrak{g})$ , etc. Simple highest weight modules are denoted by the same notation  $V(\lambda)$ .

**1.2. Crystal.** Let us review the notion of crystals briefly. See [13, 16] for detail.

**Definition 1.3.** A crystal  $\mathcal{B}$  associated with a root datum in §1.1 is a set together with maps  $\text{wt}: \mathcal{B} \rightarrow P$ ,  $\varepsilon_k, \varphi_k: \mathcal{B} \rightarrow \mathbb{Z} \sqcup \{-\infty\}$ ,  $\tilde{e}_k, \tilde{f}_k: \mathcal{B} \rightarrow \mathcal{B} \sqcup \{0\}$  ( $k \in I$ ) satisfying the following properties

$$(1.4a) \quad \varphi_k(b) = \varepsilon_k(b) + \langle h_k, \text{wt}(b) \rangle,$$

$$(1.4b) \quad \text{wt}(\tilde{e}_k b) = \text{wt}(b) + \alpha_k, \quad \varepsilon_k(\tilde{e}_k b) = \varepsilon_k(b) - 1, \quad \varphi_k(\tilde{e}_k b) = \varphi_k(b) + 1, \quad \text{if } \tilde{e}_k b \in \mathcal{B},$$

$$(1.4c) \quad \text{wt}(\tilde{f}_k b) = \text{wt}(b) - \alpha_k, \quad \varepsilon_k(\tilde{f}_k b) = \varepsilon_k(b) + 1, \quad \varphi_k(\tilde{f}_k b) = \varphi_k(b) - 1, \quad \text{if } \tilde{f}_k b \in \mathcal{B},$$

$$(1.4d) \quad b' = \tilde{f}_k b \iff b = \tilde{e}_k b' \quad \text{for } b, b' \in \mathcal{B},$$

$$(1.4e) \quad \text{if } \varphi_k(b) = -\infty \text{ for } b \in \mathcal{B}, \text{ then } \tilde{e}_k b = \tilde{f}_k b = 0$$

We set  $\text{wt}_k(b) = \langle h_k, \text{wt}(b) \rangle$ .

The crystal was introduced by abstracting the notion of crystal bases constructed by Kashiwara [13]. Thus we have the following examples of crystals.

**Notation 1.5.** (1) Let  $\mathcal{B}(\infty)$  denote the crystal associated with  $\mathbf{U}_q(\mathfrak{g})^-$ .

(2) For  $\lambda \in P^+$ , let  $\mathcal{B}(\lambda)$  denote the crystal associated with the simple  $\mathbf{U}_q(\mathfrak{g})$ -module  $V(\lambda)$  with highest weight  $\lambda$ .

We also have the following examples.

**Example 1.6.** (1) For all  $k \in I$ , we define the crystal  $\mathcal{B}_k$  as follows:

$$\begin{aligned} \mathcal{B}_k &= \{b_k(n) \mid n \in \mathbb{Z}\}, \\ \text{wt}(b_k(n)) &= n\alpha_k, \quad \varphi_k(b_k(n)) = n, \quad \varepsilon_k(b_k(n)) = -n, \\ \varphi_l(b_k(n)) &= \varepsilon_l(b_k(n)) = -\infty \quad (l \neq k), \\ \tilde{e}_k(b_k(n)) &= b_k(n+1), \quad \tilde{f}_k(b_k(n)) = b_k(n-1), \\ \tilde{e}_l(b_k(n)) &= \tilde{f}_l(b_k(n)) = 0 \quad (l \neq k). \end{aligned}$$

(2) For  $\lambda \in P^+$ , we define the crystal  $T_\lambda$  by

$$\begin{aligned} T_\lambda &= \{t_\lambda\}, \\ \text{wt}(t_\lambda) &= \lambda, \quad \varphi_k(t_\lambda) = \varepsilon_k(t_\lambda) = -\infty, \\ \tilde{e}_k(t_\lambda) &= \tilde{f}_k(t_\lambda) = 0. \end{aligned}$$

A crystal  $\mathcal{B}$  is called *normal* if

$$\varepsilon_k(b) = \max\{n \mid \tilde{e}_k^n b \neq 0\}, \quad \varphi_k(b) = \max\{n \mid \tilde{f}_k^n b \neq 0\}.$$

It is known that  $\mathcal{B}(\lambda)$  is normal.

For given two crystals  $\mathcal{B}_1, \mathcal{B}_2$ , a *morphism*  $\psi$  of crystal from  $\mathcal{B}_1$  to  $\mathcal{B}_2$  is a map  $\mathcal{B}_1 \sqcup \{0\} \rightarrow \mathcal{B}_2 \sqcup \{0\}$  satisfying  $\psi(0) = 0$  and the following conditions for all  $b \in \mathcal{B}_1, k \in I$ :

$$(1.7a) \quad \text{wt}(\psi(b)) = \text{wt}(b), \quad \varepsilon_k(\psi(b)) = \varepsilon_k(b), \quad \varphi_k(\psi(b)) = \varphi_k(b) \quad \text{if } \psi(b) \in \mathcal{B}_2,$$

$$(1.7b) \quad \tilde{e}_k \psi(b) = \psi(\tilde{e}_k b) \quad \text{if } \psi(b) \in \mathcal{B}_2, \tilde{e}_k b \in \mathcal{B}_1,$$

$$(1.7c) \quad \tilde{f}_k \psi(b) = \psi(\tilde{f}_k b) \quad \text{if } \psi(b) \in \mathcal{B}_2, \tilde{f}_k b \in \mathcal{B}_1.$$

A morphism  $\psi$  is called *strict* if  $\psi$  commutes with  $\tilde{e}_k, \tilde{f}_k$  for all  $k \in I$  without any restriction. A morphism  $\psi$  is called an *embedding* if  $\psi$  is an injective map from  $\mathcal{B}_1 \sqcup \{0\}$  to  $\mathcal{B}_2 \sqcup \{0\}$ .

**Definition 1.8.** The *tensor product*  $\mathcal{B}_1 \otimes \mathcal{B}_2$  of crystals  $\mathcal{B}_1$  and  $\mathcal{B}_2$  is defined to be the set  $\mathcal{B}_1 \times \mathcal{B}_2$  with maps defined by

$$(1.9a) \quad \text{wt}(b_1 \otimes b_2) = \text{wt}(b_1) + \text{wt}(b_2),$$

$$(1.9b) \quad \varepsilon_k(b_1 \otimes b_2) = \max(\varepsilon_k(b_1), \varepsilon_k(b_2) - \text{wt}_k(b_1)),$$

$$(1.9c) \quad \varphi_k(b_1 \otimes b_2) = \max(\varphi_k(b_2), \varphi_k(b_1) + \text{wt}_k(b_2)),$$

$$(1.9d) \quad \tilde{e}_k(b_1 \otimes b_2) = \begin{cases} \tilde{e}_k b_1 \otimes b_2 & \text{if } \varphi_k(b_1) \geq \varepsilon_k(b_2), \\ b_1 \otimes \tilde{e}_k b_2 & \text{otherwise,} \end{cases}$$

$$(1.9e) \quad \tilde{f}_k(b_1 \otimes b_2) = \begin{cases} \tilde{f}_k b_1 \otimes b_2 & \text{if } \varphi_k(b_1) > \varepsilon_k(b_2), \\ b_1 \otimes \tilde{f}_k b_2 & \text{otherwise.} \end{cases}$$

Here  $(b_1, b_2)$  is denoted by  $b_1 \otimes b_2$  and  $0 \otimes b_2$ ,  $b_1 \otimes 0$  are identified with 0.

It is easy to check that these satisfy the axioms in Definition 1.3. It is also easy to check that the tensor product of two normal crystals is again normal.

It is easy to check  $(B_1 \otimes B_2) \otimes B_3 = B_1 \otimes (B_2 \otimes B_3)$ . We denote it by  $B_1 \otimes B_2 \otimes B_3$ . Similarly we can define  $B_1 \otimes \cdots \otimes B_n$ . Then using  $B_1 \otimes \cdots \otimes B_n = (B_1 \otimes \cdots \otimes B_{n-1}) \otimes B_n$ , we can determine the tensor product of more than two crystals inductively as follows. For given  $b_1 \otimes \cdots \otimes b_n$ , we define

$$(1.10) \quad \varepsilon_k^p \stackrel{\text{def.}}{=} \varepsilon_k(b_p) - \sum_{q: q < p} \text{wt}_k(b_q), \quad \varphi_k^p \stackrel{\text{def.}}{=} \varphi_k(b_p) + \sum_{q: q > p} \text{wt}_k(b_q).$$

It is easy to show the following by induction:

$$(1.11a) \quad \text{wt}(b_1 \otimes \cdots \otimes b_n) = \sum_p \text{wt}(b_p),$$

$$(1.11b) \quad \varepsilon_k(b_1 \otimes \cdots \otimes b_n) = \max_{1 \leq p \leq n} \varepsilon_k^p,$$

$$(1.11c) \quad \varphi_k(b_1 \otimes \cdots \otimes b_n) = \max_{1 \leq p \leq n} \varphi_k^p,$$

$$(1.11d) \quad \begin{aligned} \tilde{e}_k(b_1 \otimes \cdots \otimes b_n) &= b_1 \otimes \cdots \otimes \tilde{e}_k b_p \otimes \cdots \otimes b_n \\ &\text{where } p = \min\{q \mid \varepsilon_k^q = \varepsilon_k(b_1 \otimes \cdots \otimes b_n)\}, \end{aligned}$$

$$(1.11e) \quad \begin{aligned} \tilde{f}_k(b_1 \otimes \cdots \otimes b_n) &= b_1 \otimes \cdots \otimes \tilde{f}_k b_p \otimes \cdots \otimes b_n \\ &\text{where } p = \max\{q \mid \varphi_k^q = \varphi_k(b_1 \otimes \cdots \otimes b_n)\}. \end{aligned}$$

**Definition 1.12.** A crystal  $\mathcal{B}$  is said to be of *highest weight*  $\lambda$  if the following conditions are satisfied:

- (1) there exists  $b_\lambda \in \mathcal{B}$  with  $\text{wt}(b_\lambda) = \lambda$  such that  $\tilde{e}_k(b_\lambda) = 0$  for all  $k \in I$ ,
- (2)  $\mathcal{B}$  is generated by  $b_\lambda$ , i.e., any element in  $\mathcal{B}$  is obtained from  $b_\lambda$  by applying  $\tilde{f}_k$  successively.

Note that  $b_\lambda$  is unique if it exists.

**Definition 1.13.** Suppose that a family  $\{\mathcal{D}(\lambda) \mid \lambda \in P^+\}$  of highest weight normal crystals  $\mathcal{D}(\lambda)$  of highest weight  $\lambda$  with  $b_\lambda \in \mathcal{D}(\lambda)$  satisfying the above properties is given. It is called *closed* if the crystal generated by  $b_\lambda \otimes b_\mu$  in  $\mathcal{D}(\lambda) \otimes \mathcal{D}(\mu)$  is isomorphic to  $\mathcal{D}(\lambda + \mu)$ .

We have the following characterization of  $\mathcal{B}(\lambda)$  in 1.5.

**Proposition 1.14** ([11, 6.4.21]). *If  $\{\mathcal{D}(\lambda) \mid \lambda \in P^+\}$  is a closed family of highest weight normal crystals, then  $\mathcal{D}(\lambda)$  is isomorphic to  $\mathcal{B}(\lambda)$  as a crystal for any  $\lambda \in P^+$ .*

**1.3. Quantum loop algebra.** We briefly recall the notion of quantum loop algebra. See [3, 32] for detail.

Suppose that a root datum  $P$ ,  $P^*$ , etc as in §1.1 is given. Let  $\mathbf{Lg}$  be the loop algebra  $\mathfrak{g} \otimes_{\mathbb{Q}} [z, z^{-1}]$  of the symmetrizable Kac-Moody Lie algebra  $\mathfrak{g}$ . We define the *quantum loop algebra*  $\mathbf{U}_q(\mathbf{Lg})$  as a  $\mathbb{Q}(q)$ -algebra generated by  $e_{k,r}, f_{k,r}$  ( $k \in I$ ,  $r \in \mathbb{Z}$ ),  $q^h$  ( $h \in P^*$ ),  $h_{k,m}$  ( $k \in I$ ,  $m \in \mathbb{Z} \setminus \{0\}$ ) with the following defining relations

$$(1.15a) \quad q^0 = 1, \quad q^h q^{h'} = q^{h+h'}, \quad [q^h, h_{k,m}] = 0, \quad [h_{k,m}, h_{l,n}] = 0,$$

$$(1.15b) \quad q^h e_{k,r} q^{-h} = q^{\langle h, \alpha_k \rangle} e_{k,r}, \quad q^h f_{k,r} q^{-h} = q^{-\langle h, \alpha_k \rangle} f_{k,r},$$

$$(1.15c) \quad (z - q^{\pm \langle h_l, \alpha_k \rangle} w) \psi_k^s(z) x_l^\pm(w) = (q^{\pm \langle h_l, \alpha_k \rangle} z - w) x_l^\pm(w) \psi_k^s(z),$$

$$(1.15d) \quad [x_k^+(z), x_l^-(w)] = \frac{\delta_{kl}}{q_k - q_k^{-1}} \left\{ \delta \left( \frac{w}{z} \right) \psi_k^+(w) - \delta \left( \frac{z}{w} \right) \psi_k^-(z) \right\},$$

$$(1.15e) \quad (z - q^{\pm 2} w) x_k^\pm(z) x_k^\pm(w) = (q^{\pm 2} z - w) x_k^\pm(w) x_k^\pm(z),$$

$$(1.15f) \quad \prod_{p=1}^{-\langle \alpha_k, h_l \rangle} (z - q^{\pm(b'-2p)} w) x_k^\pm(z) x_l^\pm(w) = \prod_{p=1}^{-\langle \alpha_k, h_l \rangle} (q^{\pm(b'-2p)} z - w) x_l^\pm(w) x_k^\pm(z), \quad \text{if } k \neq l,$$

$$(1.15g) \quad \sum_{\sigma \in S_b} \sum_{p=0}^b (-1)^p \begin{bmatrix} b \\ p \end{bmatrix}_{q_k} x_k^\pm(z_{\sigma(1)}) \cdots x_k^\pm(z_{\sigma(p)}) x_l^\pm(w) x_k^\pm(z_{\sigma(p+1)}) \cdots x_k^\pm(z_{\sigma(b)}) = 0, \quad \text{if } k \neq l,$$

where  $s = \pm$ ,  $b = 1 - \langle h_k, \alpha_l \rangle$ ,  $b' = -(\alpha_k, \alpha_l)$ , and  $S_b$  is the symmetric group of  $b$  letters. Here  $\delta(z)$ ,  $x_k^+(z)$ ,  $x_k^-(z)$ ,  $\psi_k^\pm(z)$  are generating functions defined by

$$\delta(z) \stackrel{\text{def.}}{=} \sum_{r=-\infty}^{\infty} z^r, \quad x_k^+(z) \stackrel{\text{def.}}{=} \sum_{r=-\infty}^{\infty} e_{k,r} z^{-r}, \quad x_k^-(z) \stackrel{\text{def.}}{=} \sum_{r=-\infty}^{\infty} f_{k,r} z^{-r},$$

$$\psi_k^\pm(z) \stackrel{\text{def.}}{=} t_k^\pm \exp \left( \pm (q_k - q_k^{-1}) \sum_{m=1}^{\infty} h_{k,\pm m} z^{\mp m} \right).$$

We also need the following generating function

$$p_k^\pm(z) \stackrel{\text{def.}}{=} \exp \left( - \sum_{m=1}^{\infty} \frac{h_{k,\pm m}}{[m]_{q_k}} z^{\mp m} \right).$$

We have  $\psi_k^\pm(z) = t_k^\pm p_k^\pm(q_k z) / p_k^\pm(q_k^{-1} z)$ .

There is a homomorphism  $\mathbf{U}_q(\mathfrak{g}) \rightarrow \mathbf{U}_q(\mathbf{Lg})$  defined by

$$q^h \mapsto q^h, \quad e_k \mapsto e_{k,0}, \quad f_k \mapsto f_{k,0}.$$

Let  $\mathbf{U}_q(\mathbf{Lg})^+$  (resp.  $\mathbf{U}_q(\mathbf{Lg})^-$ ) be the  $\mathbb{Q}(q)$ -subalgebra of  $\mathbf{U}_q(\mathbf{Lg})$  generated by elements  $e_{k,r}$ 's (resp.  $f_{k,r}$ 's). Let  $\mathbf{U}_q(\mathbf{Lg})^0$  be the  $\mathbb{Q}(q)$ -subalgebra generated by elements  $q^h$ ,  $h_{k,m}$ . We have  $\mathbf{U}_q(\mathbf{Lg}) = \mathbf{U}_q(\mathbf{Lg})^+ \cdot \mathbf{U}_q(\mathbf{Lg})^0 \cdot \mathbf{U}_q(\mathbf{Lg})^-$ .

Let  $e_{k,r}^{(n)} \stackrel{\text{def.}}{=} e_{k,r}^n / [n]_{q_k}!$ ,  $f_{k,r}^{(n)} \stackrel{\text{def.}}{=} f_{k,r}^n / [n]_{q_k}!$ . Let  $\mathbf{U}_q^{\mathbb{Z}}(\mathbf{Lg})$  be the  $\mathbb{Z}[q, q^{-1}]$ -subalgebra generated by  $e_{k,r}^{(n)}$ ,  $f_{k,r}^{(n)}$  and  $q^h$  for  $k \in I$ ,  $r \in \mathbb{Z}$ ,  $h \in P^*$ . The specialization  $\mathbf{U}_q^{\mathbb{Z}}(\mathbf{Lg}) \otimes_{\mathbb{Z}[q, q^{-1}]} \mathbb{C}$  with respect to the homomorphism  $\mathbb{Z}[q, q^{-1}] \ni q \mapsto \varepsilon \in \mathbb{C}^*$  is denoted by  $\mathbf{U}_\varepsilon(\mathbf{Lg})$  for  $\varepsilon \in \mathbb{C}^*$ .

Let  $\mathbf{U}_q^{\mathbb{Z}}(\mathbf{Lg})^+$  (resp.  $\mathbf{U}_q^{\mathbb{Z}}(\mathbf{Lg})^-$ ) be  $\mathbb{Z}[q, q^{-1}]$ -subalgebra generated by  $e_{k,r}^{(n)}$  (resp.  $f_{k,r}^{(n)}$ ) for  $k \in I$ ,  $r \in \mathbb{Z}$ ,  $n \in \mathbb{Z}_{>0}$ . Let  $\mathbf{U}_q^{\mathbb{Z}}(\mathbf{Lg})^0$  be the  $\mathbb{Z}[q, q^{-1}]$ -subalgebra generated by  $q^h$ , the coefficients of  $p_k^\pm(z)$  and

$$\begin{bmatrix} q^{h_k}; n \\ r \end{bmatrix} \stackrel{\text{def.}}{=} \prod_{s=1}^r \frac{t_k q_k^{n-s+1} - t_k^{-1} q_k^{-n+s-1}}{q_k^s - q_k^{-s}}$$

for all  $h \in P$ ,  $k \in I$ ,  $n \in \mathbb{Z}$ ,  $r \in \mathbb{Z}_{>0}$ . We have  $\mathbf{U}_q^{\mathbb{Z}}(\mathbf{Lg}) = \mathbf{U}_q^{\mathbb{Z}}(\mathbf{Lg})^+ \cdot \mathbf{U}_q^{\mathbb{Z}}(\mathbf{Lg})^0 \cdot \mathbf{U}_q^{\mathbb{Z}}(\mathbf{Lg})^-$  if  $\mathfrak{g}$  is of finite type ([3, 6.1]).

Suppose that a datum  $(\lambda, (\Psi_k^\pm(z))_{k \in I}) \in P \times \mathbb{Q}(q)[[z^\mp]]^I$  satisfying

$$\Psi_k^+(\infty) = (\alpha_k, \alpha_k) \langle \lambda, h_k \rangle / 2, \quad \Psi_k^-(0) = -(\alpha_k, \alpha_k) \langle \lambda, h_k \rangle / 2$$

is given. We say a  $\mathbf{U}_q(\mathbf{Lg})$ -module  $M$  is an  $l$ -highest weight module (' $l$ ' stands for the loop) with  $l$ -highest weight  $(\lambda, (\Psi_k^\pm(z))_{k \in I})$  if there exists a vector  $m_0 \in M$  such that

$$(1.16a) \quad e_{k,r} m_0 = 0, \quad \mathbf{U}_q(\mathbf{Lg})^- m_0 = M,$$

$$(1.16b) \quad q^h m_0 = q^{\langle h, \lambda \rangle} m_0 \quad \text{for } h \in P^*, \quad \psi_k^\pm(z) m_0 = \Psi_k^\pm(z) m_0 \quad \text{for } k \in I.$$

A  $\mathbf{U}_q(\mathbf{Lg})$ -module  $M$  is said to be  $l$ -integrable if the following two conditions are satisfied.

(a)  $M$  has a weight space decomposition as a  $\mathbf{U}_q(\mathfrak{g})$ -module:

$$M = \bigoplus_{\mu \in P} M_\mu, \quad M_\mu \stackrel{\text{def.}}{=} \{m \mid q^h \cdot v = q^{\langle h, \mu \rangle} m \text{ for any } h \in P^*\}.$$

And  $\dim M_\mu < \infty$ .

(b) For any  $m \in M$ , there exists  $n_0 \geq 1$  such that  $e_{k,r_1} \cdots e_{k,r_n} * m = f_{k,r_1} \cdots f_{k,r_n} * m = 0$  for all  $r_1, \dots, r_n \in \mathbb{Z}$ ,  $k \in I$  and  $n \geq n_0$ .

We say  $(\lambda, (\Psi_k^\pm(z))_{k \in I})$  is  $l$ -dominant, if  $\lambda \in P^+$  and there exists a  $I$ -tuple of polynomials  $P(u) = (P_k(u))_k \in \mathbb{Q}(q)[u]^I$  with  $P_k(0) = 1$  satisfying

$$(1.17) \quad \Psi_k^\pm(z) = q_k^{\deg P_k} \left( \frac{P_k(q_k^{-1}/z)}{P_k(q_k/z)} \right)^\pm,$$

where  $(\ )^\pm \in \mathbb{Q}(q)[[z^\mp]]$  denotes the expansion at  $z = \infty$  and 0 respectively.

The simple  $l$ -highest weight module  $M$  is  $l$ -integrable if and only if its  $l$ -highest weight  $(\lambda, (\Psi_k^\pm(z))_{k \in I})$  is  $l$ -dominant, provided  $\mathfrak{g}$  is of finite type ([2]) or  $\mathfrak{g}$  is symmetric ([32]). In this case,  $P(u)$  is called a *Drinfeld polynomial* of  $M$ . Since the simple  $l$ -highest weight module is determined by  $\lambda$  and  $P$ , we denote it by  $L(\lambda, P)$ .

Let  $M$  be a  $\mathbf{U}_q(\mathbf{Lg})$ -module with the weight space decomposition  $M = \bigoplus_{\mu \in P} M_\mu$  as a  $\mathbf{U}_q(\mathfrak{g})$ -module such that  $\dim M_\mu < \infty$ . Since the commutative subalgebra  $\mathbf{U}_q(\mathbf{Lg})^0$  preserves each  $M_\mu$ , we can further decompose  $M$  into a sum of generalized simultaneous eigenspaces for  $\mathbf{U}_q(\mathbf{Lg})^0$ :

$$(1.18) \quad M = \bigoplus M_{\Psi^\pm},$$

where  $\Psi^\pm(z)$  is a pair  $(\mu, (\Psi_k^\pm(z))_k)$  and

$$M_{\Psi^\pm} \stackrel{\text{def.}}{=} \left\{ m \in M \left| \begin{array}{l} q^h * m = q^{\langle h, \mu \rangle} m \text{ for } h \in P^* \\ (\psi_k^\pm(z) - \Psi_k^\pm(z) \text{Id})^N m = 0 \text{ for } k \in I \text{ and sufficiently large } N \end{array} \right. \right\}.$$

If  $M_{\Psi^\pm} \neq 0$ , we call  $M_{\Psi^\pm}$  an  $l$ -weight space, and the corresponding  $\Psi^\pm(z)$  an  $l$ -weight of  $M$ . This is a refinement of the weight space decomposition.

Let  $\lambda \in P^+$  and  $\lambda_k = \langle h_k, \lambda \rangle \in \mathbb{Z}_{\geq 0}$ . Let  $G_\lambda = \prod_{k \in I} \text{GL}(\lambda_k, \mathbb{C})$ . Its representation ring  $R(G_\lambda)$  is the invariant part of the Laurant polynomial ring:

$$R(G_\lambda) = \mathbb{Z}[x_{1,1}^\pm, \dots, x_{1,\lambda_1}^\pm]^{\mathfrak{S}_{\lambda_1}} \otimes \mathbb{Z}[x_{2,1}^\pm, \dots, x_{2,\lambda_2}^\pm]^{\mathfrak{S}_{\lambda_2}} \otimes \cdots \otimes \mathbb{Z}[x_{n,1}^\pm, \dots, x_{n,\lambda_n}^\pm]^{\mathfrak{S}_{\lambda_n}},$$



where we put a numbering  $1, \dots, n$  to  $I$ . In [32], when  $\mathfrak{g}$  is symmetric, we constructed a  $\mathbf{U}_q^{\mathbb{Z}}(\mathbf{L}\mathfrak{g}) \otimes_{\mathbb{Z}} R(G_{\lambda})$ -module  $M(\lambda)$  such that it is  $l$ -integrable and has a vector  $[0]_{\lambda}$  satisfying

$$(1.19a) \quad e_{k,r}[0]_{\lambda} = 0 \quad \text{for any } k \in I, r \in \mathbb{Z},$$

$$(1.19b) \quad M(\lambda) = (\mathbf{U}_q^{\mathbb{Z}}(\mathbf{L}\mathfrak{g})^{-} \otimes_{\mathbb{Z}} R(G_{\lambda})) [0]_{\lambda},$$

$$(1.19c) \quad q^h[0]_{\lambda} = q^{\langle h, \lambda \rangle} [0]_{\lambda},$$

$$(1.19d) \quad \psi_k^{\pm}(z)[0]_{\lambda} = q^{w_k} \left( \prod_{i=1}^{w_k} \frac{1 - q^{-1}x_{k,i}/z}{1 - qx_{k,i}/z} \right)^{\pm} [0]_{\lambda}.$$

We call this a *universal standard module*. Its construction, via quiver varieties, will be explained briefly in §6.

If an  $I$ -tuple of monic polynomials  $P(u) = (P_k(u))_{k \in I}$  with  $\deg P_k = \langle h_k, \lambda \rangle$  are given, then we define a *standard module* by the specialization

$$M(\lambda, P) = M(\lambda) \otimes_{R(G_{\lambda})[q, q^{-1}]} \mathbb{C}(q),$$

where the algebra homomorphism  $R(G_{\lambda})[q, q^{-1}] \rightarrow \mathbb{C}(q)$  sends  $x_{k,1}, \dots, x_{k,w_k}$  to roots of  $P_k$ . The simple module  $L(\lambda, P)$  is the simple quotient of  $M(\lambda, P)$ .

It is known that  $M(\lambda)$  is free as an  $R(G_{\lambda})$ -module ([32, 7.3.5]). Thus  $M(\lambda, P)$  depends on  $P$  *continuously*, while  $L(\lambda, P)$  depends *discontinuously*.

**1.4. Drinfeld realization.** Now we assume that  $\mathfrak{g}$  is of finite type, i.e.,  $(\ , \ )$  is positive definite. We take a root datum so that  $\text{rank } P = \text{rank } \mathfrak{g}$ . It is well-known that the untwisted affine Lie algebra  $\widehat{\mathfrak{g}} = \mathbf{L}\mathfrak{g} \oplus \mathbb{Q}c \oplus \mathbb{Q}d$  is a Kac-Moody Lie algebra with a root datum

$$\widehat{P}^* = P^* \oplus \mathbb{Z}c \oplus \mathbb{Z}d, \quad \widehat{I} = I \cup \{0\},$$

and certain  $\alpha_0, h_0, (\ , \ )$ . The *quantum affine algebra*  $\mathbf{U}_q(\widehat{\mathfrak{g}})$  is the quantum enveloping algebra associated with this root datum. Let  $\mathbf{U}_q(\widehat{\mathfrak{g}})'$  be the subalgebra of  $\mathbf{U}_q(\widehat{\mathfrak{g}})/(q^c - 1)$  generated by  $e_k, f_k$  ( $k \in \widehat{I}$ ) and  $q^h$  ( $h \in P$ ). Then a comultiplication  $\Delta$  is defined on  $\mathbf{U}_q(\widehat{\mathfrak{g}})'$  by the same formula (1.2). The integral form  $\mathbf{U}_q^{\mathbb{Z}}(\widehat{\mathfrak{g}})'$  is defined in the same way.

In [6] Drinfeld observed that there exists an isomorphism  $\mathbf{U}_q(\mathbf{L}\mathfrak{g}) \rightarrow \mathbf{U}_q(\widehat{\mathfrak{g}})'$  of  $\mathbb{Q}(q)$ -algebras. Its explicit form was given by Beck [1]:

$$e_{k,r} = o(k)^r T_{\tilde{\omega}_k}^{-r}(e_k), \quad f_{k,r} = o(k)^r T_{\tilde{\omega}_k}^r(f_k),$$

where  $o: I \rightarrow \{\pm 1\}$  is an orientation of  $I$  such that  $o(k) = -o(l)$  if  $a_{kl} \neq 0$  for  $k \neq l$ , and  $T_{\tilde{\omega}_k}$  is an automorphism of  $\mathbf{U}_q(\widehat{\mathfrak{g}})'$  defined by Lusztig (the braid group action). Since  $T_{\tilde{\omega}_k}$  preserves  $\mathbf{U}_q^{\mathbb{Z}}(\widehat{\mathfrak{g}})'$ , the isomorphism induces an isomorphism  $\mathbf{U}_q^{\mathbb{Z}}(\mathbf{L}\mathfrak{g}) \rightarrow \mathbf{U}_q^{\mathbb{Z}}(\widehat{\mathfrak{g}})'$ .

We shall identify  $\mathbf{U}_q(\widehat{\mathfrak{g}})'$  with  $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$  when  $\mathfrak{g}$  is of finite type hereafter. In particular, we have a comultiplication  $\Delta$  on  $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$ . We need the following asymptotic formula, which can be deduced from [5, 10].

**Lemma 1.20.** (1) *On finite dimensional  $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$ -modules, we have*

$$\Delta(h_{k,\pm m}) = h_{k,\pm m} \otimes 1 + 1 \otimes h_{k,\pm m} + a \text{ nilpotent term.}$$

(2) *Let  $V$  and  $W$  be finite dimensional  $\mathbf{U}_q(\mathbf{L}\mathfrak{g})$ -modules. Suppose that  $V$  has a vector  $b$  such that  $e_{k,r}b = 0$  for all  $k \in I, r \in \mathbb{Z}$ . Then a subspace  $\{b\} \otimes W \subset V \otimes W$  is invariant under  $\mathbf{U}_q(\mathbf{L}\mathfrak{g})^+$ , and the map  $W \ni x \mapsto b \otimes x \in \{b\} \otimes W$  respects  $\mathbf{U}_q(\mathbf{L}\mathfrak{g})^+$ -module structure.*

*Remark 1.21.* The property (2) depends on our choice of the comultiplication (1.2). If we take one in [20], then the property holds after we exchange  $V$  and  $W$ .

Before we close this section, we give an algebraic characterization of the standard module. We assume  $\mathfrak{g}$  is of type  $ADE$ . For each fundamental weight  $\Lambda_k$ , the universal standard module  $M(\Lambda_k)$  is a  $\mathbf{U}_q^{\mathbb{Z}}(\mathbf{L}\mathfrak{g}) \otimes_{\mathbb{Z}[q, q^{-1}]} R(\mathbb{C}^* \times \mathbb{C}^*) \cong \mathbf{U}_q^{\mathbb{Z}}(\mathbf{L}\mathfrak{g})[x, x^{-1}]$ -module. We set  $W(\Lambda_k) = M(\Lambda_k)/(x-1)M(\Lambda_k)$ . It is called an *l-fundamental representation*.

**Theorem 1.22.** *Put a numbering  $1, \dots, n$  on  $I$ . Let  $\lambda_k = \langle h_k, \lambda \rangle$ . The universal standard module  $M(\lambda)$  is the submodule of*

$$W(\Lambda_1)^{\otimes \lambda_1} \otimes \cdots \otimes W(\Lambda_n)^{\otimes \lambda_n} \otimes \mathbb{Z}[q, q^{-1}, x_{1,1}^{\pm}, \dots, x_{1,\lambda_1}^{\pm}, \dots, x_{n,1}^{\pm}, \dots, x_{n,\lambda_n}^{\pm}]^{\mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_n}}$$

(the tensor product is over  $\mathbb{Z}[q, q^{-1}]$ ) generated by  $\bigotimes_{k \in I} [0]_{\Lambda_k}^{\otimes \lambda_k}$ . (The result holds for the tensor product of any order.)

This theorem was not explicitly stated in [32], but can be deduced by above properties as follows: First we make tensor products of both modules by the quotient field of  $R(G_\lambda)[q, q^{-1}]$ . Then both modules are simple and have the same Drinfeld polynomial (cf. proof of Lemma 6.4). Thus there exists a unique isomorphism from  $M(\lambda) \otimes (\text{quotient field})$  to the tensor product, sending  $[0]_\lambda$  to  $\bigotimes_{k \in I} [0]_{\Lambda_k}^{\otimes \lambda_k}$ . Now by the property (1.19b) and the freeness of  $M(\lambda)$ , we have the assertion.

*Remark 1.23.* Varagnolo-Vasserot [37, §7] conjectured that  $M(\lambda)$  is isomorphic to a module studied by Kashiwara [15] ( $V(\lambda)$  in his notation), after tensoring  $\mathbb{Q}$  and forgetting the symmetric group invariance. When  $\lambda$  is a fundamental weight, both modules are isomorphic, since they are simple and have the same Drinfeld polynomial. Kashiwara conjectures that his module  $V(\lambda)$  has the property in Theorem 1.22 [loc. cit., §13]. Thus Varagnolo-Vasserot's conjecture is equivalent to Kashiwara's conjecture. On the other hand, Kashiwara shows that the submodule above has a global crystal base [loc. cit., Theorem 8.5]. ( $N_{\mathbb{Q}}$  in his notation.) Probably, this base coincides with the (conjectural) canonical base considered in [25] as analogue of [22, 23].

## 2. PRELIMINARIES (II) – GEOMETRIC PART

In this paper, all varieties are defined over  $\mathbb{C}$ .

**2.1.  $K$ -homology groups.** Let  $X$  be a quasi-projective variety. Its integral Borel-Moore homology group of degree  $k$  is denoted by  $H_k(X, \mathbb{Z})$ . Set  $H_*(X, \mathbb{Z}) = \bigoplus_k H_k(X, \mathbb{Z})$ . When a linear algebraic group  $G$  acts algebraically on  $X$ , we denote by  $K^G(X)$  the Grothendieck group of the abelian category of  $G$ -equivariant coherent sheaves on  $X$  (the  $K$ -homology group). It is a module over  $R(G)$ , the representation ring of  $G$ . We shall use several operations on  $H_*(X, \mathbb{Z})$  and  $K^G(X)$  in this article, but we do not review them here. See [32, §6] and [4].

**2.2. Quiver variety.** We briefly review the notion of quiver varieties. The reference to results can be found in [28, §2], unless referred explicitly.

Suppose a root datum is given. We assume that it is symmetric, i.e.,  $(\alpha_k, \alpha_k) = 2$  for all  $k \in I$ . Then the generalized Cartan matrix  $\mathbf{C}$  is equal to  $((\alpha_k, \alpha_l))_{k,l \in I}$  and symmetric. To the root datum, we associate a finite graph  $(I, E)$  as follows (the Dynkin diagram). The set of vertices is identified with  $I$ , and we draw  $(\alpha_k, \alpha_l)$  edges between vertices  $k$  and  $l$  ( $k \neq l$ ). We give no edge loops, edge joining a vertex with itself. Conversely a finite graph without edge loops gives a *symmetric* generalized Cartan matrix.

Let  $H$  be the set of pairs consisting of an edge together with its orientation. For  $h \in H$ , we denote by  $\text{in}(h)$  (resp.  $\text{out}(h)$ ) the incoming (resp. outgoing) vertex of  $h$ . For  $h \in H$  we denote by  $\bar{h}$  the same edge as  $h$  with the reverse orientation. Choose and fix a numbering  $1, 2, \dots, n$  on  $I$ . We then define a subset  $\Omega \subset H$  so that  $h \in \Omega$  if  $\text{out}(h) < \text{in}(h)$ . Then  $\Omega$  is an orientation of the graph, i.e.,  $\bar{\Omega} \cup \Omega = H$ ,  $\Omega \cap \bar{\Omega} = \emptyset$ . (The numbering and the orientation will not play a role until §8.) The pair  $(I, \Omega)$  is called a *quiver*.

If  $V^1 = \bigoplus_k V_k^1$  and  $V^2 = \bigoplus_k V_k^2$  are  $I$ -graded vector spaces, we define vector spaces by

$$(2.1) \quad L(V^1, V^2) \stackrel{\text{def.}}{=} \bigoplus_{k \in I} \text{Hom}(V_k^1, V_k^2), \quad E(V^1, V^2) \stackrel{\text{def.}}{=} \bigoplus_{h \in H} \text{Hom}(V_{\text{out}(h)}^1, V_{\text{in}(h)}^2)$$

For  $B = (B_h) \in E(V^1, V^2)$  and  $C = (C_h) \in E(V^2, V^3)$ , let us define a multiplication of  $B$  and  $C$  by

$$CB \stackrel{\text{def.}}{=} \left( \sum_{\text{in}(h)=k} C_h B_{\bar{h}} \right)_k \in L(V^1, V^3).$$

Multiplications  $ba$ ,  $Ba$  of  $a \in L(V^1, V^2)$ ,  $b \in L(V^2, V^3)$ ,  $B \in E(V^2, V^3)$  are defined in obvious manner. If  $a \in L(V^1, V^1)$ , its trace  $\text{tr}(a)$  is understood as  $\sum_k \text{tr}(a_k)$ .

If  $V$  and  $W$  are  $I$ -graded vector spaces, we consider the vector spaces

$$(2.2) \quad \mathbf{M} \equiv \mathbf{M}(V, W) \stackrel{\text{def.}}{=} E(V, V) \oplus L(W, V) \oplus L(V, W),$$

where we use the notation  $\mathbf{M}$  unless we want to specify  $V, W$ . The above three components for an element of  $\mathbf{M}$  is denoted by  $B, i, j$  respectively.

**Convention 2.3.** When quiver varieties will be related the representation theory, we will choose  $V$  and  $W$  corresponding to a pair  $(\mathbf{v}, \mathbf{w}) \in Q^+ \times P^+$ . The rule is  $\dim V_k = v_k$ ,  $\dim W_k = \langle h_k, \mathbf{w} \rangle$ , where  $\mathbf{v} = \sum_k v_k \alpha_k$ . Conversely  $V$  determines  $\mathbf{v}$ , while  $W$  determines  $\mathbf{w}$  modulo an element  $*$  such that  $\langle h_k, * \rangle = 0$  for all  $h_k$ . But the action of  $\mathfrak{g}$  on simple highest weight modules  $V(\mathbf{w})$  and  $V(\mathbf{w} + *)$  differ only by scalars (see [12, 9.10]). So essentially there is no ambiguity.

For an  $I$ -graded subspace  $S = \bigoplus_k S_k$  of subspaces  $V$  and  $B \in E(V, V)$ , we say  $S$  is *B-invariant* if  $B_h(S_{\text{out}(h)}) \subset S_{\text{in}(h)}$ .

Fix a function  $\varepsilon: H \rightarrow \mathbb{C}^*$  such that  $\varepsilon(h) + \varepsilon(\bar{h}) = 0$  for all  $h \in H$ . For  $B \in E(V^1, V^2)$ , let us denote by  $\varepsilon B \in E(V^1, V^2)$  data given by  $(\varepsilon B)_h = \varepsilon(h) B_h$  for  $h \in H$ .

Let us define a symplectic form  $\omega$  on  $\mathbf{M}$  by

$$(2.4) \quad \omega((B, i, j), (B', i', j')) \stackrel{\text{def.}}{=} \text{tr}(\varepsilon B B') + \text{tr}(ij' - i'j).$$

Let  $G_V \stackrel{\text{def.}}{=} \prod_k \text{GL}(V_k)$ . It acts on  $\mathbf{M}$  by

$$(2.5) \quad (B, i, j) \mapsto g \cdot (B, i, j) \stackrel{\text{def.}}{=} (gB g^{-1}, gi, jg^{-1})$$

preserving the symplectic form  $\omega$ . The moment map  $\mu: \mathbf{M} \rightarrow L(V, V)$  vanishing at the origin is given by

$$(2.6) \quad \mu(B, i, j) = \varepsilon B B + ij,$$

where the dual of the Lie algebra of  $G_V$  is identified with the Lie algebra via the trace. Let  $\mu^{-1}(0)$  be an affine algebraic variety (not necessarily irreducible) defined as the zero set of  $\mu$ .

**Definition 2.7.** A point  $(B, i, j) \in \mu^{-1}(0)$  is said to be *stable* if the following condition holds:

if an  $I$ -graded subspace  $S = \bigoplus_k S_k$  of  $V$  is  $B$ -invariant and contained in  $\text{Ker } j$ , then  $S = 0$ .

Let us denote by  $\mu^{-1}(0)^s$  the set of stable points.

Clearly, the stability condition is invariant under the action of  $G_V$ . Hence we may say an orbit is stable or not.

Let  $\mathfrak{M} \equiv \mathfrak{M}(\mathbf{v}, \mathbf{w}) \stackrel{\text{def.}}{=} \mu^{-1}(0)^s / G_V$ . We use the notation  $\mathfrak{M}$  unless we need to specify dimensions of  $V$  and  $W$ . It is known that the  $G_V$ -action is free on  $\mu^{-1}(0)^s$  and  $\mathfrak{M}$  is a nonsingular quasi-projective variety, having a symplectic form induced by  $\omega$ . A  $G_V$ -orbit though  $(B, i, j)$ , considered as a point of  $\mathfrak{M}$  is denoted by  $[B, i, j]$ . Since the action is free,  $V$  and  $W$  can be considered as  $I$ -graded vector bundles over  $\mathfrak{M}$ . We denote them by the same notation. We consider  $E(V, V)$ ,  $L(W, V)$ ,  $L(V, W)$  as vector bundles defined by the same formula as in (2.1). By the definition,  $B, i, j$  can be considered as sections of those bundles.

Let us consider three-term sequence of vector bundles over  $\mathfrak{M}(\mathbf{v}, \mathbf{w})$  given by

$$(2.8) \quad L(V, V) \xrightarrow{\iota} E(V, V) \oplus L(W, V) \oplus L(V, W) \xrightarrow{d\mu} L(V, V),$$

where  $d\mu$  is the differential of  $\mu$  at  $(B, i, j)$ , i.e.,

$$d\mu(C, I, J) = \varepsilon BC + \varepsilon CB + iJ + Ij,$$

and  $\iota$  is given by

$$\iota(\xi) = (B\xi - \xi B) \oplus (-\xi i) \oplus j\xi.$$

Then  $\iota$  is injective and  $d\mu$  is surjective, and the tangent bundle of  $\mathfrak{M}(\mathbf{v}, \mathbf{w})$  is identified with  $\text{Ker } d\mu / \text{Im } \iota$ .

Let  $\mathfrak{M}_0 \equiv \mathfrak{M}_0(\mathbf{v}, \mathbf{w}) \stackrel{\text{def.}}{=} \mu^{-1}(0) // G_V$ , where  $//$  is the affine algebro-geometric quotient, i.e., the coordinate ring of  $\mu^{-1}(0) // G_V$  is  $G_V$ -invariant polynomials on  $\mu^{-1}(0)$ . It is an affine algebraic variety, and identified with the set of closed  $G_V$ -orbits in  $\mu^{-1}(0)$  as a set.

There exists a projective morphism  $\pi: \mathfrak{M} \rightarrow \mathfrak{M}_0$ , sending  $[B, i, j]$  to the unique closed orbit contained in the closure of the orbit  $G_V \cdot (B, i, j)$ .

Let  $\mathfrak{L} \equiv \mathfrak{L}(\mathbf{v}, \mathbf{w}) \stackrel{\text{def.}}{=} \pi^{-1}(0)$ . It is a lagrangian subvariety in  $\mathfrak{M}(\mathbf{v}, \mathbf{w})$ .

If  $\mathbf{v}' - \mathbf{v} \in Q^+$ , then  $\mathfrak{M}_0(\mathbf{v}, \mathbf{w})$  can be identified with a closed subvariety of  $\mathfrak{M}_0(\mathbf{v}', \mathbf{w})$ . We consider the direct limit  $\mathfrak{M}_0(\infty, \mathbf{w}) \stackrel{\text{def.}}{=} \bigcup_{\mathbf{v}} \mathfrak{M}_0(\mathbf{v}, \mathbf{w})$ . If the graph is of finite type,  $\mathfrak{M}_0(\mathbf{v}, \mathbf{w})$  stabilizes at some  $\mathbf{v}$ . This is *not* true in general. However, it has no harm in this paper. We use  $\mathfrak{M}_0(\infty, \mathbf{w})$  to simplify the notation, and do not need any structures on it. We can always work on individual  $\mathfrak{M}_0(\mathbf{v}, \mathbf{w})$ , not on  $\mathfrak{M}_0(\infty, \mathbf{w})$ .

We set  $\mathfrak{M}(\mathbf{w}) \stackrel{\text{def.}}{=} \bigsqcup_{\mathbf{v}} \mathfrak{M}(\mathbf{v}, \mathbf{w})$ ,  $\mathfrak{L}(\mathbf{w}) \stackrel{\text{def.}}{=} \bigsqcup_{\mathbf{v}} \mathfrak{L}(\mathbf{v}, \mathbf{w})$ . They may have infinitely many components, but no harm as above.

Let  $\Delta(\mathbf{v}, \mathbf{w})$  denote the diagonal in  $\mathfrak{M}(\mathbf{v}, \mathbf{w}) \times \mathfrak{M}(\mathbf{v}, \mathbf{w})$ . It is a lagrangian subvariety, if we endow a symplectic form  $\omega \times (-\omega)$  on  $\mathfrak{M}(\mathbf{v}, \mathbf{w}) \times \mathfrak{M}(\mathbf{v}, \mathbf{w})$ .

For  $n \in \mathbb{Z}_{>0}$ , we define  $\mathfrak{P}_k^{(n)}(\mathbf{v}, \mathbf{w})$  by

$$(2.9) \quad \mathfrak{P}_k^{(n)}(\mathbf{v}, \mathbf{w}) \stackrel{\text{def.}}{=} \{(B, i, j, S) \mid (B, i, j) \in \mathbf{M}(V, W), S \subset V \text{ as below}\} / G_V,$$

- (a)  $(B, i, j) \in \mu^{-1}(0)^s$ ,
- (b)  $S$  is a  $B$ -invariant subspace containing the image of  $i$  with  $\dim S = \mathbf{v} - n\alpha_k$ .

When  $n = 1$ , we simply denote it by  $\mathfrak{P}_k(\mathbf{v}, \mathbf{w})$ . If we set  $\mathbf{v}' = \mathbf{v} - n\alpha_k$ , we have a natural morphism  $\mathfrak{P}_k^{(n)}(\mathbf{v}, \mathbf{w}) \rightarrow \mathfrak{M}(\mathbf{v}', \mathbf{w}) \times \mathfrak{M}(\mathbf{v}, \mathbf{w})$  by

$$[B, i, j, S] \mapsto ([B', i', j'], [B, i, j]),$$

where  $(B', i', j')$  is the restriction of  $(B, i, j)$  to  $S$ .

Then  $\mathfrak{P}_k^{(n)}(\mathbf{v}, \mathbf{w})$  is a nonsingular closed lagrangian subvariety in  $\mathfrak{M}(\mathbf{v}', \mathbf{w}) \times \mathfrak{M}(\mathbf{v}, \mathbf{w})$  ([32, 11.2.3]). The quotient  $V/V'$  defines a rank  $n$  vector bundle over  $\mathfrak{P}_k^{(n)}(\mathbf{v}, \mathbf{w})$ .

Let  $G_W = \prod_{k \in I} \mathrm{GL}(W_k)$ . It acts naturally on  $\mathbf{M}$ ,  $\mathfrak{M}$  and  $\mathfrak{M}_0$ . We define  $\mathbb{C}^*$ -actions on  $\mathfrak{M}$  and  $\mathfrak{M}_0$  by

$$B_h \mapsto t^{m(h)+1} B_h, \quad i \mapsto ti, \quad j \mapsto tj \quad (t \in \mathbb{C}^*),$$

where  $m: H \rightarrow \mathbb{Z}$  is a certain function determined by a numbering on edges joining common vertices (see [32, 2.7]). When the root datum is simply-laced, i.e., symmetric and  $(\alpha_k, \alpha_l) \in \{0, 1\}$  for  $k \neq l$ , we have  $m \equiv 0$ . We denote this action by  $(B, i, j) \mapsto g * (B, i, j)$  or  $[B, i, j] \mapsto g * [B, i, j]$  for  $g \in G_W \times \mathbb{C}^*$ . The vector bundles  $V, W, E(V, V), L(W, V), L(V, W)$  are  $G_W \times \mathbb{C}^*$ -equivariant bundles, and  $B, i, j$  are equivariant sections.

Let  $L(m)$  be the 1-dimensional  $\mathbb{C}^*$ -module defined by  $t \mapsto t^m$  for  $m \in \mathbb{Z}$ . For a  $\mathbb{C}^*$ -module  $V$ ,  $L(m) \otimes V$  is denoted by  $q^m V$ . That is  $L(1)$  is identified with  $q$ .

We consider the following  $G_W \times \mathbb{C}^*$ -equivariant complex  $C_k^\bullet$  over  $\mathfrak{M}$ :

$$(2.10) \quad C_k^\bullet \equiv C_k^\bullet(\mathbf{v}, \mathbf{w}): q^{-1} V_k \xrightarrow{\sigma_k} \bigoplus_{l: k \neq l} [-\langle h_k, \alpha_l \rangle]_q V_l \oplus W_k \xrightarrow{\tau_k} q V_k,$$

where

$$\sigma_k = \bigoplus_{\mathrm{in}(h)=k} B_h \oplus j_k, \quad \tau_k = \sum_{\mathrm{in}(h)=k} \varepsilon(h) B_h + i_k.$$

We assign degree 0 to the middle term.

Let  $Q_k(\mathbf{v}, \mathbf{w})$  the degree 0 cohomology of the complex (2.10), i.e.,

$$Q_k(\mathbf{v}, \mathbf{w}) \stackrel{\mathrm{def.}}{=} \mathrm{Ker} \tau_k / \mathrm{Im} \sigma_k.$$

We introduce the following subsets of  $\mathfrak{M}(\mathbf{v}, \mathbf{w})$ :

$$(2.11) \quad \mathfrak{M}_{k;n}(\mathbf{v}, \mathbf{w}) \stackrel{\mathrm{def.}}{=} \left\{ [B, i, j] \in \mathfrak{M}(\mathbf{v}, \mathbf{w}) \mid \mathrm{codim}_{V_k} \mathrm{Im} \tau_k = n \right\}$$

$$\mathfrak{M}_{k;\leq n}(\mathbf{v}, \mathbf{w}) \stackrel{\mathrm{def.}}{=} \bigcup_{m \leq n} \mathfrak{M}_{k;m}(\mathbf{v}, \mathbf{w}), \quad \mathfrak{M}_{k;\geq n}(\mathbf{v}, \mathbf{w}) \stackrel{\mathrm{def.}}{=} \bigcup_{m \geq n} \mathfrak{M}_{k;m}(\mathbf{v}, \mathbf{w}).$$

Since  $\mathfrak{M}_{k;\leq n}(\mathbf{v}, \mathbf{w})$  is an open subset of  $\mathfrak{M}(\mathbf{v}, \mathbf{w})$ ,  $\mathfrak{M}_{k;n}(\mathbf{v}, \mathbf{w})$  is a locally closed subvariety. The restriction of  $Q_k(\mathbf{v}, \mathbf{w})$  to  $\mathfrak{M}_{k;n}(\mathbf{v}, \mathbf{w})$  is a  $G_W \times \mathbb{C}^*$ -equivariant vector bundle of rank  $\langle h_k, \mathbf{w} - \mathbf{v} \rangle + n$ .

Replacing  $V_k$  by  $\mathrm{Im} \tau_k$ , we have a natural map

$$(2.12) \quad p: \mathfrak{M}_{k;n}(\mathbf{v}, \mathbf{w}) \rightarrow \mathfrak{M}_{k;0}(\mathbf{v} - n\alpha_k, \mathbf{w}).$$

Note that the projection  $\pi: \mathfrak{M}(\mathbf{v}, \mathbf{w}) \rightarrow \mathfrak{M}_0(\mathbf{v}, \mathbf{w})$  factors through  $p$ . In particular, the fiber of  $\pi$  is preserved under  $p$ .

**Proposition 2.13.** *Let  $G(n, Q_k(\mathbf{v} - n\alpha_k, \mathbf{w})|_{\mathfrak{M}_{k;0}(\mathbf{v} - n\alpha_k, \mathbf{w})})$  be the Grassmann bundle of  $n$ -planes in the vector bundle obtained by restricting  $Q_k(\mathbf{v} - n\alpha_k, \mathbf{w})$  to  $\mathfrak{M}_{k;0}(\mathbf{v} - n\alpha_k, \mathbf{w})$ . Then we have the following diagram:*

$$\begin{array}{ccc} G(n, Q_k(\mathbf{v} - n\alpha_k, \mathbf{w})|_{\mathfrak{M}_{k;0}(\mathbf{v} - n\alpha_k, \mathbf{w})}) & \xrightarrow{\Pi} & \mathfrak{M}_{k;0}(\mathbf{v} - n\alpha_k, \mathbf{w}) \\ \downarrow \cong & & \parallel \\ \mathfrak{M}_{k;n}(\mathbf{v}, \mathbf{w}) & \xrightarrow{p} & \mathfrak{M}_{k;0}(\mathbf{v} - n\alpha_k, \mathbf{w}), \end{array}$$

where  $\Pi$  is the natural projection. The kernel of the natural surjective homomorphism  $p^*Q_k(\mathbf{v} - n\alpha_k, \mathbf{w}) \rightarrow Q_k(\mathbf{v}, \mathbf{w})$  is isomorphic to the tautological vector bundle of the Grassmann bundle of the first row.

The following formula will play an important role later.

$$(2.14) \quad \dim \text{fiber of } p = n(\langle h_k, \mathbf{w} - \mathbf{v} \rangle + n) = \frac{1}{2} (\dim \mathfrak{M}(\mathbf{v}, \mathbf{w}) - \dim \mathfrak{M}(\mathbf{v} - n\alpha_k, \mathbf{w})).$$

### 3. VARIETIES $\mathfrak{Z}$ , $\tilde{\mathfrak{Z}}$ AND THEIR EQUIVARIANT $K$ -THEORIES

The main body of this article starts from this section. For the sake of space, we only consider the case of tensor products of *two* modules. The arguments can be generalized to the case of  $N$  modules in a straightforward way. We will mention in §7.

Let  $\mathbf{w}, \mathbf{w}^1, \mathbf{w}^2 \in P^+$  be dominant weights such that  $\mathbf{w} = \mathbf{w}^1 + \mathbf{w}^2$ . These will be fixed until §7.

Let us fix a direct sum decomposition  $W = W^1 \oplus W^2$  of  $I$ -graded vector spaces with  $\langle h_k, \mathbf{w}^1 \rangle = \dim W_k^1$ ,  $\langle h_k, \mathbf{w}^2 \rangle = \dim W_k^2$ . Set  $G_{W^1} = \prod_k \text{GL}(W_k^1)$ ,  $G_{W^2} = \prod_k \text{GL}(W_k^2)$ .

We define a three-term sequence of vector bundles over  $\mathfrak{M}(\mathbf{v}^1, \mathbf{w}^1) \times \mathfrak{M}(\mathbf{v}^2, \mathbf{w}^2)$  by

$$(3.1) \quad L(V^1, V^2) \xrightarrow{\alpha^{21}} E(V^1, V^2) \oplus L(W^1, V^2) \oplus q L(V^1, W^2) \xrightarrow{\beta^{21}} L(V^1, V^2),$$

where

$$\begin{aligned} \alpha^{21}(\xi) &= (B^2\xi - \xi B^1) \oplus (-\xi i^1) \oplus j^2\xi, \\ \beta^{21}(C \oplus I \oplus J) &= \varepsilon B^2C + \varepsilon C B^1 + i^2J + I j^1. \end{aligned}$$

This is a complex, that is  $\beta^{21}\alpha^{21} = 0$ , thanks to the equation  $\varepsilon B^p B^p + i^p j^p = 0$  ( $p = 1, 2$ ). It is  $\mathbb{C}^* \times G_{W^1} \times G_{W^2}$ -equivariant.

By the same argument as in [30, 3.10],  $\alpha^{21}$  is injective and  $\beta^{21}$  is surjective. Thus the quotient  $\text{Ker } \beta^{21} / \text{Im } \alpha^{21}$  is a vector bundle over  $\mathfrak{M}(\mathbf{v}^1, \mathbf{w}^1) \times \mathfrak{M}(\mathbf{v}^2, \mathbf{w}^2)$  with rank  $(\mathbf{v}^1, \mathbf{w}^2) + (\mathbf{w}^1, \mathbf{v}^2) - (\mathbf{v}^1, \mathbf{v}^2)$ .

We define a one parameter subgroup  $\lambda: \mathbb{C}^* \rightarrow G_W$  by

$$\lambda(t) = \text{id}_{W^1} \oplus t \text{id}_{W^2} \in G_{W^1} \times G_{W^2} \subset G_W.$$

**Lemma 3.2.** *The fixed point set of  $\lambda(\mathbb{C}^*)$  in  $\mathfrak{M}(\mathbf{v}, \mathbf{w})$  is isomorphic to  $\bigsqcup_{\mathbf{v}^1 + \mathbf{v}^2 = \mathbf{v}} \mathfrak{M}(\mathbf{v}^1, \mathbf{w}^1) \times \mathfrak{M}(\mathbf{v}^2, \mathbf{w}^2)$ .*

*Proof.* A point  $[B, i, j] \in \mathfrak{M}(\mathbf{v}, \mathbf{w})$  is fixed by  $\lambda(\mathbb{C}^*)$  if and only if there exists a one parameter subgroup  $\rho: \mathbb{C}^* \rightarrow G_V$  such that

$$\lambda(t) * (B, i, j) = \rho(t)^{-1} \cdot (B, i, j).$$

Let  $V^1$  (resp.  $V^2$ ) be the eigenspace of  $V$  with eigenvalue 1 (resp.  $t$ ). Let  $V'$  be the sum of other eigenspaces. The above equation implies that

- (1)  $B(V^1) \subset V^1, B(V^2) \subset V^2, B(V') \subset V',$
- (2)  $i(W^1) \subset V^1, i(W^2) \subset V^2,$
- (3)  $j(V^1) \subset W^1, j(V^2) \subset W^2, j(V') = 0.$

The stability condition implies that  $V' = 0$ . Thus we have  $V = V^1 \oplus V^2$ , and  $[B, i, j]$  decomposes into a sum  $[B^1, i^1, j^1] \in \mathfrak{M}(\mathbf{v}^1, \mathbf{w}^1)$  and  $[B^2, i^2, j^2] \in \mathfrak{M}(\mathbf{v}^2, \mathbf{w}^2)$ , where  $\mathbf{v}^1 = \dim V^1$ ,  $\mathbf{v}^2 = \dim V^2$ .

Conversely  $([B^1, i^1, j^1], [B^2, i^2, j^2]) \in \mathfrak{M}(\mathbf{v}^1, \mathbf{w}^1) \times \mathfrak{M}(\mathbf{v}^2, \mathbf{w}^2)$  defines a fixed point. Thus we have a surjective morphism

$$\bigsqcup_{\mathbf{v}^1 + \mathbf{v}^2 = \mathbf{v}} \mathfrak{M}(\mathbf{v}^1, \mathbf{w}^1) \times \mathfrak{M}(\mathbf{v}^2, \mathbf{w}^2) \rightarrow \mathfrak{M}(\mathbf{v}, \mathbf{w})^{\lambda(\mathbb{C}^*)}.$$

By the freeness of the  $G_V$ -action on  $\mu^{-1}(0)^s$ , this is injective.

Let us identify the tangent bundle of  $\mathfrak{M}(\mathbf{v}, \mathbf{w})$  with  $\text{Ker } d\mu / \text{Im } \iota$  in (2.8). Its restriction to the fixed point set  $\mathfrak{M}(\mathbf{v}^1, \mathbf{w}^1) \times \mathfrak{M}(\mathbf{v}^2, \mathbf{w}^2)$  decomposes as

$$(\text{Ker } d\mu^1 / \text{Im } \iota^1) \oplus (\text{Ker } d\mu^2 / \text{Im } \iota^2) \oplus (\text{Ker } \beta^{21} / \text{Im } \alpha^{21}) \oplus (\text{Ker } \beta^{12} / \text{Im } \alpha^{12}),$$

where  $\iota^p$ ,  $d\mu^p$  are as in (2.8) with  $\mathfrak{M}(\mathbf{v}, \mathbf{w})$  replaced by  $\mathfrak{M}(\mathbf{v}^p, \mathbf{w}^p)$  ( $p = 1, 2$ ),  $\alpha^{21}$ ,  $\beta^{21}$  are as above, and  $\alpha^{12}$ ,  $\beta^{12}$  are defined by exchanging the role of  $V^1, W^1$  and  $V^2, W^2$ . We let  $\lambda(\mathbb{C}^*)$  acts on  $V^1$  (resp.  $V^2$ ) with weight 0 (resp. 1). Then this identification respects the  $\lambda(\mathbb{C}^*)$ -action. Thus the tangent space of the fixed point component, which is the 0-weight space of the whole tangent space, is identified with  $(\text{Ker } d\mu^1 / \text{Im } \iota^1) \oplus (\text{Ker } d\mu^2 / \text{Im } \iota^2)$ . It is isomorphic to the tangent space of  $\mathfrak{M}(\mathbf{v}^1, \mathbf{w}^1) \times \mathfrak{M}(\mathbf{v}^2, \mathbf{w}^2)$ . Therefore the map is isomorphism.  $\square$

We move all  $\mathbf{v}, \mathbf{v}^1, \mathbf{v}^2$ . We get

$$\mathfrak{M}(\mathbf{w}^1) \times \mathfrak{M}(\mathbf{w}^2) \cong \mathfrak{M}(\mathbf{w})^{\lambda(\mathbb{C}^*)}.$$

There are a natural  $R(G_{W^1} \times G_{W^2} \times \mathbb{C}^*)$ -homomorphism

$$(3.3) \quad \boxtimes: K^{G_{W^1} \times \mathbb{C}^*}(\mathfrak{M}(\mathbf{v}^1, \mathbf{w}^1)) \otimes_{R(\mathbb{C}^*)} K^{G_{W^2} \times \mathbb{C}^*}(\mathfrak{M}(\mathbf{v}^2, \mathbf{w}^2)) \ni (E, F) \\ \longmapsto E \boxtimes F \in K^{G_{W^1} \times G_{W^2} \times \mathbb{C}^*}(\mathfrak{M}(\mathbf{v}^1, \mathbf{w}^1) \times \mathfrak{M}(\mathbf{v}^2, \mathbf{w}^2))$$

and a similar homomorphisms for  $\mathfrak{L}(\mathbf{v}^1, \mathbf{w}^1)$  and  $\mathfrak{L}(\mathbf{v}^2, \mathbf{w}^2)$ . We call them *Künneth homomorphisms*.

**Theorem 3.4.** *The two Künneth homomorphisms are isomorphisms. If  $A$  is an abelian reductive subgroup of  $G_{W^1} \times G_{W^2} \times \mathbb{C}^*$ , the same holds for the fixed point set  $\mathfrak{M}(\mathbf{v}^1, \mathbf{w}^1)^A \times \mathfrak{M}(\mathbf{v}^2, \mathbf{w}^2)^A$ ,  $\mathfrak{L}(\mathbf{v}^1, \mathbf{w}^1)^A \times \mathfrak{L}(\mathbf{v}^2, \mathbf{w}^2)^A$ .*

*Proof.* In [32, §7], the following was shown: varieties  $\mathfrak{M}(\mathbf{v}, \mathbf{w})$ ,  $\mathfrak{L}(\mathbf{v}, \mathbf{w})$  and fixed point sets have  $\alpha$ -partitions (see [loc. cit., 7.1] for definition) such that each piece is an affine space bundle over a nonsingular projective manifold, which is a locally equivariant vector bundle. Moreover, each base manifold has a decomposable diagonal class as in [loc. cit., 7.2.1].

Note that Künneth homomorphisms are defined for arbitrary varieties, and have obvious functorial properties. By the arguments in [loc. cit., §7.1], it is enough to show that the Künneth homomorphism is an isomorphism for each base manifold. By the proof of [loc. cit., 7.2.1], each base manifold has this property.  $\square$

Let us define subsets of  $\mathfrak{M}(\mathbf{w})$  by

$$\mathfrak{Z} \equiv \mathfrak{Z}(\mathbf{w}^1; \mathbf{w}^2) \stackrel{\text{def.}}{=} \left\{ [B, i, j] \in \mathfrak{M}(\mathbf{w}) \mid \lim_{t \rightarrow 0} \lambda(t) * [B, i, j] \text{ exists} \right\},$$

$$\tilde{\mathfrak{Z}} \equiv \tilde{\mathfrak{Z}}(\mathbf{w}^1; \mathbf{w}^2) \stackrel{\text{def.}}{=} \left\{ [B, i, j] \in \mathfrak{M}(\mathbf{w}) \mid \lim_{t \rightarrow 0} \lambda(t) * [B, i, j] \in \mathfrak{L}(\mathbf{w}^1) \times \mathfrak{L}(\mathbf{w}^2) \right\}.$$

We use the symbol  $\mathfrak{Z}(\mathbf{w}^1; \mathbf{w}^2)$ ,  $\tilde{\mathfrak{Z}}(\mathbf{w}^1; \mathbf{w}^2)$  when we want to emphasize the dimensions. These subsets are invariant under the action of  $G_{W^1} \times G_{W^2} \times \mathbb{C}^*$ .

Since  $\pi: \mathfrak{M}(\mathbf{w}) \rightarrow \mathfrak{M}_0(\mathbf{w})$  is a projective morphism, the following is clear.

**Lemma 3.5.** *We have*

$$\mathfrak{Z} = \left\{ [B, i, j] \in \mathfrak{M}(\mathbf{w}) \mid \lim_{t \rightarrow 0} \lambda(t) * \pi([B, i, j]) \text{ exists} \right\},$$

$$\tilde{\mathfrak{Z}} = \left\{ [B, i, j] \in \mathfrak{M}(\mathbf{w}) \mid \lim_{t \rightarrow 0} \lambda(t) * \pi([B, i, j]) = 0 \right\}.$$

In particular,  $\mathfrak{Z}, \tilde{\mathfrak{Z}}$  are  $\pi$ -saturated, i.e.,  $\pi^{-1}(\pi(\mathfrak{Z})) = \mathfrak{Z}$ ,  $\pi^{-1}(\pi(\tilde{\mathfrak{Z}})) = \tilde{\mathfrak{Z}}$ .

**Lemma 3.6.**  *$\mathfrak{Z}$  and  $\tilde{\mathfrak{Z}}$  are closed subvarieties of  $\mathfrak{M}(\mathbf{w})$ .*

*Proof.* It is enough to show that  $\pi(\mathfrak{Z})$  is a closed subvariety of  $\mathfrak{M}_0(\infty, \mathbf{w})$ . By [21, 1.3] the coordinate ring  $\mathfrak{M}_0(\mathbf{v}, \mathbf{w})$  (the ring of  $G_V$ -invariant polynomials in  $\mu^{-1}(0) \subset \mathbf{M}(V, W)$ ) is generated by the following two types of functions:

- (a)  $\text{tr}(B_{h_N} B_{h_{N-1}} \cdots B_{h_1} : V_{\text{out}(h_1)} \rightarrow V_{\text{out}(h_1)})$ , where  $h_1, \dots, h_N$  is a cycle in our graph, i.e.,  $\text{in}(h_1) = \text{out}(h_2)$ ,  $\text{in}(h_2) = \text{out}(h_3)$ ,  $\dots$ ,  $\text{in}(h_{N-1}) = \text{out}(h_N)$ ,  $\text{in}(h_N) = \text{out}(h_1)$ .
- (b)  $\chi(j_{\text{in}(h_N)} B_{h_N} B_{h_{N-1}} \cdots B_{h_1} i_{\text{out}(h_1)})$ , where  $h_1, \dots, h_N$  is a path in our graph, i.e.,  $\text{in}(h_1) = \text{out}(h_2)$ ,  $\text{in}(h_2) = \text{out}(h_3)$ ,  $\dots$ ,  $\text{in}(h_{N-1}) = \text{out}(h_N)$ , and  $\chi$  is a linear form on  $\text{Hom}(W_{\text{out}(h_1)}, W_{\text{in}(h_N)})$ .

Functions of the first type are invariant under the  $\lambda(\mathbb{C}^*)$ -action. Functions of the second type are of weight 1,  $-1$ , 0 if  $\chi$  is the extension (by 0) of a linear form of  $\text{Hom}(W_{\text{out}(h_1)}^1, W_{\text{in}(h_N)}^2)$ ,  $\text{Hom}(W_{\text{out}(h_1)}^2, W_{\text{in}(h_N)}^1)$ ,  $\text{Hom}(W_{\text{out}(h_1)}^1, W_{\text{in}(h_N)}^1) \oplus \text{Hom}(W_{\text{out}(h_1)}^2, W_{\text{in}(h_N)}^2)$  respectively.

Thus  $[B, i, j] \in \mathfrak{M}_0(\mathbf{v}, \mathbf{w})$  is contained in  $\pi(\mathfrak{Z})$  if and only if  $j_{\text{in}(h_N)} B_{h_N} B_{h_{N-1}} \cdots B_{h_1} i_{\text{out}(h_1)}$  maps  $W_{\text{out}(h_1)}^2$  into  $W_{\text{in}(h_N)}^2$  for any path  $h_1, \dots, h_N$ . Similarly  $[B, i, j] \in \mathfrak{M}_0(\mathbf{v}, \mathbf{w})$  is contained in  $\pi(\tilde{\mathfrak{Z}})$  if and only if functions of the first type vanishes, and  $j_{\text{in}(h_N)} B_{h_N} B_{h_{N-1}} \cdots B_{h_1} i_{\text{out}(h_1)}$  maps  $W_{\text{out}(h_1)}^2$  into 0, and  $W_{\text{out}(h_1)}^1$  to  $W_{\text{in}(h_N)}^2$  for any path  $h_1, \dots, h_N$ . Now the assertions are clear from these descriptions.  $\square$

The limit  $\lim_{t \rightarrow 0} \lambda(t) * [B, i, j]$  must be contained in the fixed point set  $\mathfrak{M}(\mathbf{w})^{\lambda(\mathbb{C}^*)}$  if it exists. Thus we have the decomposition

$$\mathfrak{Z}(\mathbf{w}^1; \mathbf{w}^2) = \bigsqcup_{\mathbf{v}^1, \mathbf{v}^2} \mathfrak{Z}(\mathbf{v}^1, \mathbf{w}^1; \mathbf{v}^2, \mathbf{w}^2),$$

$$\text{where } \mathfrak{Z}(\mathbf{v}^1, \mathbf{w}^1; \mathbf{v}^2, \mathbf{w}^2) \stackrel{\text{def.}}{=} \left\{ [B, i, j] \mid \lim_{t \rightarrow 0} \lambda(t) * [B, i, j] \in \mathfrak{M}(\mathbf{v}^1, \mathbf{w}^1) \times \mathfrak{M}(\mathbf{v}^2, \mathbf{w}^2) \right\}.$$

Similarly we have  $\tilde{\mathfrak{Z}}(\mathbf{w}^1; \mathbf{w}^2) = \bigcup_{\mathbf{v}^1, \mathbf{v}^2} \tilde{\mathfrak{Z}}(\mathbf{v}^1, \mathbf{w}^1; \mathbf{v}^2, \mathbf{w}^2)$  defined exactly in the same way. These are the Bialynicki-Birula decomposition of  $\mathfrak{Z}, \tilde{\mathfrak{Z}}$ . Thus we have



**Proposition 3.7.** (1)  $\mathfrak{Z}(\mathbf{v}^1, \mathbf{w}^1; \mathbf{v}^2, \mathbf{w}^2)$ ,  $\tilde{\mathfrak{Z}}(\mathbf{v}^1, \mathbf{w}^1; \mathbf{v}^2, \mathbf{w}^2)$  are nonsingular locally closed subvarieties of  $\mathfrak{M}(\mathbf{v}^1 + \mathbf{v}^2, \mathbf{w}^1 + \mathbf{w}^2)$ .

(2) The map

$$\mathfrak{Z}(\mathbf{v}^1, \mathbf{w}^1; \mathbf{v}^2, \mathbf{w}^2) \ni [B, i, j] \longmapsto \lim_{t \rightarrow 0} \lambda(t) * [B, i, j] \in \mathfrak{M}(\mathbf{v}^1, \mathbf{w}^1) \times \mathfrak{M}(\mathbf{v}^2, \mathbf{w}^2)$$

identifies  $\mathfrak{Z}(\mathbf{v}^1, \mathbf{w}^1; \mathbf{v}^2, \mathbf{w}^2)$  with a fiber bundle over  $\mathfrak{M}(\mathbf{v}^1, \mathbf{w}^1) \times \mathfrak{M}(\mathbf{v}^2, \mathbf{w}^2)$ , where the fiber over  $([B^1, i^1, j^1], [B^2, i^2, j^2])$  is isomorphic to the affine space given by the direct sum of eigenspaces with positive weights in the tangent space  $T_{([B^1, i^1, j^1], [B^2, i^2, j^2])} \mathfrak{M}(\mathbf{v}^1 + \mathbf{v}^2, \mathbf{w}^1 + \mathbf{w}^2)$ . Similarly the map  $\tilde{\mathfrak{Z}}(\mathbf{v}^1, \mathbf{w}^1; \mathbf{v}^2, \mathbf{w}^2) \rightarrow \mathfrak{L}(\mathbf{v}^1, \mathbf{w}^1) \times \mathfrak{L}(\mathbf{v}^2, \mathbf{w}^2)$  is a fiber bundle with the same fiber. Both fiber bundles are locally  $G_{W^1} \times G_{W^2} \times \mathbb{C}^*$ -equivariant vector bundles.

(3) There exists an ordering  $<$  on the set  $\{(\mathbf{v}^1, \mathbf{v}^2) \mid \mathbf{v}^1 + \mathbf{v}^2 = \mathbf{v}\}$  (the set of components of  $\mathfrak{M}(\mathbf{v}, \mathbf{w})^{\lambda(\mathbb{C}^*)}$ ) such that

$$\bigcup_{(\mathbf{v}^1, \mathbf{v}^2) \leq (\mathbf{v}_0^1, \mathbf{v}_0^2)} \mathfrak{Z}(\mathbf{v}^1, \mathbf{w}^1; \mathbf{v}^2, \mathbf{w}^2), \quad \left( \text{resp.} \quad \bigcup_{(\mathbf{v}^1, \mathbf{v}^2) \leq (\mathbf{v}_0^1, \mathbf{v}_0^2)} \tilde{\mathfrak{Z}}(\mathbf{v}^1, \mathbf{w}^1; \mathbf{v}^2, \mathbf{w}^2) \right)$$

are closed subvarieties of  $\mathfrak{Z}$  (resp.  $\tilde{\mathfrak{Z}}$ ) for any fixed  $(\mathbf{v}_0^1, \mathbf{v}_0^2)$ .

See [32, 7.2.5]. (The assumption on the properness of the moment map, which does not hold in the present case, was used to ensure that the whole space  $X$  is a union of  $+$ -attracting sets.)

*Remark 3.8.* In fact, the order  $<$  in (3) can be described explicitly. If  $f$  is the moment map, then

$$(\mathbf{v}^1, \mathbf{v}^2) < (\mathbf{v}'^1, \mathbf{v}'^2) \implies f(\mathfrak{M}(\mathbf{v}^1, \mathbf{w}^1) \times \mathfrak{M}(\mathbf{v}^2, \mathbf{w}^2)) > f(\mathfrak{M}(\mathbf{v}'^1, \mathbf{w}^1) \times \mathfrak{M}(\mathbf{v}'^2, \mathbf{w}^2)).$$

With the Kähler metric in [28], we have

$$f(\mathfrak{M}(\mathbf{v}^1, \mathbf{w}^1) \times \mathfrak{M}(\mathbf{v}^2, \mathbf{w}^2)) = \sum_{k \in I} \zeta_{\mathbb{R}}^{(k)} \dim V_k^2,$$

where  $\zeta_{\mathbb{R}}^{(k)} \in \mathbb{R}_{>0}$  ( $k \in I$ ) are parameter for the Kähler metric. The ordering  $<$  is independent of the metric, so we can move the parameters. Therefore we may assume  $(\mathbf{v}^1, \mathbf{v}^2) \leq (\mathbf{v}'^1, \mathbf{v}'^2)$  if and only if  $\dim V_k^2 \geq \dim V_k'^2$  for all  $k \in I$ . In particular,  $(0, \mathbf{v})$  is a minimal element if  $\mathfrak{M}(0, \mathbf{w}^1) \times \mathfrak{M}(\mathbf{v}, \mathbf{w}^2)$  is nonempty. Thus  $\mathfrak{Z}(0, \mathbf{w}^1; \mathbf{v}, \mathbf{w}^2)$  and  $\tilde{\mathfrak{Z}}(0, \mathbf{w}^1; \mathbf{v}, \mathbf{w}^2)$  are (nonsingular) closed subvarieties. These are analogue of  $\mathfrak{Z}_1, \tilde{\mathfrak{Z}}_1$  in [23, 7.10], and will play an important role later.

Proposition 3.7 has the following corollary.

**Theorem 3.9.** (1) Both  $\mathfrak{Z}$  and  $\tilde{\mathfrak{Z}}$  satisfy the properties (S) and  $(T_{G_{W^1} \times G_{W^2} \times \mathbb{C}^*})$ . (See [32, §7.1] for the definition.)

(2) We have an exact sequence

$$0 \rightarrow K^{G_{W^1} \times G_{W^2} \times \mathbb{C}^*}(\mathfrak{Z}_{\leq (\mathbf{v}_0^1, \mathbf{v}_0^2)}) \rightarrow K^{G_{W^1} \times G_{W^2} \times \mathbb{C}^*}(\mathfrak{Z}) \rightarrow K^{G_{W^1} \times G_{W^2} \times \mathbb{C}^*}(\mathfrak{Z} \setminus \mathfrak{Z}_{\leq (\mathbf{v}_0^1, \mathbf{v}_0^2)}) \rightarrow 0,$$

where  $\mathfrak{Z}_{\leq (\mathbf{v}_0^1, \mathbf{v}_0^2)} \stackrel{\text{def.}}{=} \bigcup_{(\mathbf{v}^1, \mathbf{v}^2) \leq (\mathbf{v}_0^1, \mathbf{v}_0^2)} \mathfrak{Z}(\mathbf{v}^1, \mathbf{w}^1; \mathbf{v}^2, \mathbf{w}^2)$ . The same holds if we replace  $\mathfrak{Z}$  by  $\tilde{\mathfrak{Z}}$ .

(3) The direct image maps

$$K^{G_{W^1} \times G_{W^2} \times \mathbb{C}^*}(\mathfrak{L}(\mathbf{w})) \rightarrow K^{G_{W^1} \times G_{W^2} \times \mathbb{C}^*}(\tilde{\mathfrak{Z}}) \rightarrow K^{G_{W^1} \times G_{W^2} \times \mathbb{C}^*}(\mathfrak{Z}) \rightarrow K^{G_{W^1} \times G_{W^2} \times \mathbb{C}^*}(\mathfrak{M}(\mathbf{w}))$$

(induced by the inclusions  $\mathfrak{L}(\mathbf{w}) \subset \tilde{\mathfrak{Z}} \subset \mathfrak{Z} \subset \mathfrak{M}(\mathbf{w})$ ) are injective. All four modules are free of the same rank.

*Proof.* (Compare [23, 6.17].) (1)(2) The assertion follows from Proposition 3.7 and results in [32, §7].

(3) We replace the group  $G_{W^1} \times G_{W^2} \times \mathbb{C}^*$  by its maximal torus  $H_{\mathbf{w}^1} \times H_{\mathbf{w}^2} \times \mathbb{C}^*$ . If we tensor the fraction field of  $R(H_{\mathbf{w}^1} \times H_{\mathbf{w}^2} \times \mathbb{C}^*)$  to the above homomorphisms, it becomes isomorphisms by [36] since the  $H_{\mathbf{w}^1} \times H_{\mathbf{w}^2} \times \mathbb{C}^*$ -fixed points are the same on all four varieties. Then the assertion for the torus follows from the freeness ((1) and [32, 7.3.5]) of modules. Taking the Weyl group invariant part, we get the assertion.  $\square$

Let  $\mathfrak{Z}(\mathbf{w}^2; \mathbf{w}^1)$ ,  $\tilde{\mathfrak{Z}}(\mathbf{w}^2; \mathbf{w}^1)$  denote the varieties exactly as above except that the role of  $W^1$  and  $W^2$  are exchanged. By the description in the proof of Lemma 3.6, the intersection  $\mathfrak{Z}(\mathbf{w}^2; \mathbf{w}^1) \cap \tilde{\mathfrak{Z}}(\mathbf{w}^1; \mathbf{w}^2)$  is equal to  $\mathfrak{L}(\mathbf{w})$ . In particular, we can define a bilinear pairing by

$$(3.10) \quad K^{G_{W^1} \times G_{W^2} \times \mathbb{C}^*}(\mathfrak{Z}(\mathbf{w}^2; \mathbf{w}^1)) \times K^{G_{W^1} \times G_{W^2} \times \mathbb{C}^*}(\tilde{\mathfrak{Z}}(\mathbf{w}^1; \mathbf{w}^2)) \ni (F, F') \\ \longmapsto p_*(F \otimes_{\mathfrak{M}(\mathbf{w})}^L F') \in R(G_{W^1} \times G_{W^2} \times \mathbb{C}^*),$$

where  $p: \mathfrak{L}(\mathbf{w}) \rightarrow \text{point}$  is the projection to the point. A pairing

$$(3.11) \quad H_*(\mathfrak{Z}(\mathbf{w}^2; \mathbf{w}^1), \mathbb{Z}) \times H_*(\tilde{\mathfrak{Z}}(\mathbf{w}^1; \mathbf{w}^2), \mathbb{Z}) \rightarrow \mathbb{Z}$$

can be defined in a similar way.

**Theorem 3.12.** *Pairings (3.10), (3.11) are nondegenerate.*

The proof is the same as that in [32, §7].

We need more precise description of the projection  $\mathfrak{Z}(\mathbf{v}^1, \mathbf{w}^1; \mathbf{v}^2, \mathbf{w}^2) \rightarrow \mathfrak{M}(\mathbf{v}^1, \mathbf{w}^1) \times \mathfrak{M}(\mathbf{v}^2, \mathbf{w}^2)$  later. By the description of the tangent bundle in the proof of Lemma 3.2, the fiber of the projection is isomorphic to  $\text{Ker } \beta^{21} / \text{Im } \alpha^{21}$ , the cohomology of the complex (3.1).

Fix representatives  $(B^1, i^1, j^1)$ ,  $(B^2, i^2, j^2)$  of a point in  $\mathfrak{M}(\mathbf{v}^1, \mathbf{w}^1) \times \mathfrak{M}(\mathbf{v}^2, \mathbf{w}^2)$ . We take  $(C, I, J) \in \text{Ker } \beta^{21}$ . We define a data  $(B, i, j)$  in  $\mathbf{M}(V, W)$  so that its components in  $\mathbf{M}(V^1, W^1)$ ,  $\mathbf{M}(V^2, W^2)$ ,  $E(V^1, V^2) \oplus L(W^1, V^2) \oplus L(V^1, W^2)$  are given by  $(B^1, i^1, j^1)$ ,  $(B^2, i^2, j^2)$ ,  $(C, I, J)$  respectively. Then it satisfies  $\varepsilon BB + ij = 0$ . We claim that  $(B, i, j)$  is stable: Suppose that  $S$  is contained in  $\text{Ker } j$  and invariant under  $B$ . Under the projection  $V^1 \oplus V^2 \rightarrow V^1$ , the subspace  $S$  define a subspace  $S^1 \subset V^1$  which is contained in  $\text{Ker } j^1$  and invariant under  $B^1$ . By the stability condition for  $(B^1, i^1, j^1)$ , we have  $S^1 = 0$ . Therefore  $S \subset V^2$ . Now the stability condition for  $(B^2, i^2, j^2)$  implies that  $S = 0$ . Thus we have a morphism  $\text{Ker } \beta^{21} \rightarrow \mu^{-1}(0)^s$ . If  $(C, I, J) - (C', I', J') \in \text{Im } \alpha^{21}$ , corresponding two data are in the same  $G_V$ -orbit. Thus we have a morphism  $\text{Ker } \beta^{21} / \text{Im } \alpha^{21} \rightarrow \mathfrak{Z}(\mathbf{v}^1, \mathbf{w}^2; \mathbf{v}^2, \mathbf{w}^2)$ . (The left hand side is the total space of the vector bundle.)

**Proposition 3.13.** *The morphism*

$$\text{Ker } \beta^{21} / \text{Im } \alpha^{21} \rightarrow \mathfrak{Z}(\mathbf{v}^1, \mathbf{w}^2; \mathbf{v}^2, \mathbf{w}^2)$$

*is an isomorphism.*

*Proof.* The morphism is equivariant under the  $\lambda(\mathbb{C}^*)$ -action. Thus it is enough to check the assertion in a neighbourhood of  $\mathfrak{M}(\mathbf{v}^1, \mathbf{w}^1) \times \mathfrak{M}(\mathbf{v}^2, \mathbf{w}^2)$ . The tangent bundles of these varieties, restricted to  $\mathfrak{M}(\mathbf{v}^1, \mathbf{w}^1) \times \mathfrak{M}(\mathbf{v}^2, \mathbf{w}^2)$ , are both given by

$$(\text{Ker } \beta^{21} / \text{Im } \alpha^{21}) \oplus T(\mathfrak{M}(\mathbf{v}^1, \mathbf{w}^1) \times \mathfrak{M}(\mathbf{v}^2, \mathbf{w}^2)),$$

and the differential of the morphism is the identity. Hence we have the assertion.  $\square$

We also have an isomorphism

$$\mathrm{Ker} \beta^{21} / \mathrm{Im} \alpha^{21} |_{\mathcal{L}(\mathbf{v}^1, \mathbf{w}^2) \times \mathcal{L}(\mathbf{v}^2, \mathbf{w})} \rightarrow \tilde{\mathfrak{Z}}(\mathbf{v}^1, \mathbf{w}^2; \mathbf{v}^2, \mathbf{w}^2),$$

where the left hand side is the total space of the restriction of the vector bundle.

**Proposition 3.14.** (1) *The set of irreducible components of  $\tilde{\mathfrak{Z}}$  is naturally identified with the sets of irreducible components of  $\mathcal{L}(\mathbf{w}^1) \times \mathcal{L}(\mathbf{w}^2)$ .*

(2)  *$\tilde{\mathfrak{Z}}$  is a lagrangian subvariety of  $\mathfrak{M}(\mathbf{w})$ . More precisely,  $\tilde{\mathfrak{Z}} \cap \mathfrak{M}(\mathbf{v}, \mathbf{w})$  is a lagrangian subvariety of  $\mathfrak{M}(\mathbf{v}, \mathbf{w})$  for each  $\mathbf{v}$ .*

*Proof.* Recall that  $\tilde{\mathfrak{Z}} \cap \mathfrak{M}(\mathbf{v}, \mathbf{w})$  is a finite union  $\bigcup_{\mathbf{v}^1 + \mathbf{v}^2 = \mathbf{v}} \tilde{\mathfrak{Z}}(\mathbf{v}^1, \mathbf{w}^2; \mathbf{v}^2, \mathbf{w}^2)$ . Hence both assertions will follow if we show that  $\tilde{\mathfrak{Z}}(\mathbf{v}^1, \mathbf{w}^2; \mathbf{v}^2, \mathbf{w}^2)$  is a lagrangian subvariety.

Let  $\rho: \tilde{\mathfrak{Z}}(\mathbf{v}^1, \mathbf{w}^2; \mathbf{v}^2, \mathbf{w}^2) \rightarrow \mathcal{L}(\mathbf{v}^1, \mathbf{w}^1) \times \mathcal{L}(\mathbf{v}^2, \mathbf{w}^2)$  be the projection in Proposition 3.7(2). By Proposition 3.13, we have the following exact sequence of vector bundles over  $\tilde{\mathfrak{Z}}(\mathbf{v}^1, \mathbf{w}^2; \mathbf{v}^2, \mathbf{w}^2)$ :

$$0 \rightarrow \rho^* (\mathrm{Ker} \beta^{21} / \mathrm{Im} \alpha^{21}) \rightarrow T\tilde{\mathfrak{Z}}(\mathbf{v}^1, \mathbf{w}^2; \mathbf{v}^2, \mathbf{w}^2) \rightarrow \rho^* (T\mathcal{L}(\mathbf{v}^1, \mathbf{w}^1) \oplus T\mathcal{L}(\mathbf{v}^2, \mathbf{w}^2)) \rightarrow 0.$$

(More precisely, we must restrict to the inverse image of the nonsingular locus of  $\mathcal{L}(\mathbf{v}^1, \mathbf{w}^1) \times \mathcal{L}(\mathbf{v}^2, \mathbf{w}^2)$ .) By the definition of  $\mathrm{Ker} \beta^{21} / \mathrm{Im} \alpha^{21}$ , we have

$$\omega \left( \rho^* (\mathrm{Ker} \beta^{21} / \mathrm{Im} \alpha^{21}), T\tilde{\mathfrak{Z}}(\mathbf{v}^1, \mathbf{w}^2; \mathbf{v}^2, \mathbf{w}^2) \right) = 0,$$

and the induced bilinear form on  $\rho^* (T\mathcal{L}(\mathbf{v}^1, \mathbf{w}^1) \oplus T\mathcal{L}(\mathbf{v}^2, \mathbf{w}^2))$  coincides with one induced from the symplectic form on  $\mathfrak{M}(\mathbf{v}^1, \mathbf{w}^1) \times \mathfrak{M}(\mathbf{v}^2, \mathbf{w}^2)$ . Since  $\mathcal{L}(\mathbf{v}^1, \mathbf{w}^1)$ ,  $\mathcal{L}(\mathbf{v}^2, \mathbf{w}^2)$  are lagrangian, the latter vanishes.

It is also clear that the dimension of  $\tilde{\mathfrak{Z}}(\mathbf{v}^1, \mathbf{w}^2; \mathbf{v}^2, \mathbf{w}^2)$  is half of that of  $\mathfrak{M}(\mathbf{v}, \mathbf{w})$ . Just note that  $\mathrm{Ker} \beta^{21} / \mathrm{Im} \alpha^{21}$  and  $\mathrm{Ker} \beta^{12} / \mathrm{Im} \alpha^{12}$  are dual to each other with respect to the symplectic form.  $\square$

*Remark 3.15.* By [18],  $\mathfrak{M}(\mathbf{v}, \mathbf{w})$  can be identified with a framed moduli space of holomorphic vector bundles on an ALE space, when the graph of type *ADE*, where  $\mathbf{v}$ ,  $\mathbf{w}$  correspond to the Chern class and the framing. We have the following geometric description:

$$\mathfrak{Z}(\mathbf{v}^1, \mathbf{w}^1; \mathbf{v}^2, \mathbf{w}^2) = \{\text{an exact sequence } 0 \rightarrow E^2 \rightarrow E \rightarrow E^1 \rightarrow 0\},$$

where  $E^1$  (resp.  $E^2$ ) has the Chern class and the framing corresponding to  $\mathbf{v}^1, \mathbf{w}^1$  (resp.  $\mathbf{v}^2, \mathbf{w}^2$ ). Here the exact sequence is suppose to respect the framing. The inclusion  $\mathfrak{Z}(\mathbf{v}^1, \mathbf{w}^1; \mathbf{v}^2, \mathbf{w}^2) \rightarrow \mathfrak{M}(\mathbf{v}, \mathbf{w})$  is  $(0 \rightarrow E^2 \rightarrow E \rightarrow E^1 \rightarrow 0) \mapsto E$ , and the projection  $\mathfrak{Z}(\mathbf{v}^1, \mathbf{w}^1; \mathbf{v}^2, \mathbf{w}^2) \rightarrow \mathfrak{M}(\mathbf{v}^1, \mathbf{w}^1) \times \mathfrak{M}(\mathbf{v}^2, \mathbf{w}^2)$  is  $(0 \rightarrow E^2 \rightarrow E \rightarrow E^1 \rightarrow 0) \mapsto (E^1, E^2)$ . The fiber over  $(E^1, E^2)$  is the extension group  $\mathrm{Ext}^1(E^2, E^1)$  (for the framed vector bundle). This is nothing but  $\mathrm{Ker} \beta^{21} / \mathrm{Im} \alpha^{21}$ . The duality between  $\mathrm{Ker} \beta^{21} / \mathrm{Im} \alpha^{21}$  and  $\mathrm{Ker} \beta^{12} / \mathrm{Im} \alpha^{12}$  is nothing but the Serre duality  $\mathrm{Ext}^1(E^2, E^1)^* \cong \mathrm{Ext}^1(E^1, E^2)$ , where the canonical bundle of the ALE space is trivial.

#### 4. CRYSTAL STRUCTURE

Let  $\mathrm{Irr} \tilde{\mathfrak{Z}}$  be the set of irreducible components of  $\tilde{\mathfrak{Z}}$ . It is a disjoint union of irreducible components of  $\tilde{\mathfrak{Z}} \cap \mathfrak{M}(\mathbf{v}, \mathbf{w})$  for various  $\mathbf{v}$ .

We define  $\mathrm{wt}: \mathrm{Irr} \tilde{\mathfrak{Z}} \rightarrow P$  by setting

$$\mathrm{wt}(X) = \mathbf{w} - \mathbf{v}$$

if  $X$  is an irreducible component of  $\tilde{\mathfrak{Z}} \cap \mathfrak{M}(\mathbf{v}, \mathbf{w})$ .

Let  $X$  be an irreducible component of  $\tilde{\mathfrak{Z}}$ . Taking a generic element  $[B, i, j] \in X$ , we define

$$\varepsilon_k(X) \stackrel{\text{def.}}{=} \dim(V_k / \text{Im } \tau_k).$$

It defines a function  $\varepsilon_k: \text{Irr } \tilde{\mathfrak{Z}} \rightarrow \mathbb{Z}_{\geq 0}$ . We define  $\varphi_k: \text{Irr } \tilde{\mathfrak{Z}} \rightarrow \mathbb{Z}_{\geq 0}$  by

$$\varphi_k(X) = \varepsilon_k(X) + \langle h_k, \text{wt}(X) \rangle.$$

We set  $\text{wt}_k(X) = \langle h_k, \text{wt}(X) \rangle$  as before. We have

$$\dim(\text{Ker } \tau_k / \text{Im } \sigma_k) = \varphi_k(X).$$

Suppose  $\varepsilon_k(X) > 0$  and  $\text{wt } X = \mathbf{w} - \mathbf{v}$ . Then we have Grassmann bundles

$$\begin{aligned} p: \mathfrak{M}_{k; \varepsilon_k(X)}(\mathbf{v}, \mathbf{w}) &\rightarrow \mathfrak{M}_{k; 0}(\mathbf{v} - \varepsilon_k(X)\alpha_k, \mathbf{w}), \\ p': \mathfrak{M}_{k; \varepsilon_k(X)-1}(\mathbf{v} - \alpha_k, \mathbf{w}) &\rightarrow \mathfrak{M}_{k; 0}(\mathbf{v} - \varepsilon_k(X)\alpha_k, \mathbf{w}). \end{aligned}$$

We define an irreducible component of  $\tilde{\mathfrak{Z}} \cap \mathfrak{M}(\mathbf{v} - \alpha_k, \mathbf{w})$  by

$$X' \stackrel{\text{def.}}{=} \overline{p'^{-1}(p(X \cap \mathfrak{M}_{k; \varepsilon_k(X)}(\mathbf{v}, \mathbf{w}))}.$$

In fact,

- (1) Since  $\pi$  factors through  $p$ ,  $p'^{-1}(p(X \cap \mathfrak{M}_{k; \varepsilon_k(X)}(\mathbf{v}, \mathbf{w})))$  is contained in  $\tilde{\mathfrak{Z}}$  by Lemma 3.5.
- (2) We have  $\dim p'^{-1}(p(X \cap \mathfrak{M}_{k; \varepsilon_k(X)}(\mathbf{v}, \mathbf{w}))) = \frac{1}{2} \dim \mathfrak{M}(\mathbf{v} - \alpha_k, \mathbf{w})$  by (2.14).

Thus  $X'$  is an irreducible component of  $\tilde{\mathfrak{Z}} \cap \mathfrak{M}(\mathbf{v} - \alpha_k, \mathbf{w})$ . We define an operator  $\tilde{e}_k: \text{Irr } \tilde{\mathfrak{Z}} \rightarrow \text{Irr } \tilde{\mathfrak{Z}} \sqcup \{0\}$  by

$$(4.1) \quad \tilde{e}_k(X) \stackrel{\text{def.}}{=} \begin{cases} X' & \text{if } \varepsilon_k(X) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly if  $\varphi_k(X) > 0$ , we consider a Grassmann bundle

$$p'': \mathfrak{M}_{k; \varepsilon_k(X)+1}(\mathbf{v} + \alpha_k, \mathbf{w}) \rightarrow \mathfrak{M}_{k; 0}(\mathbf{v} - \varepsilon_k(X)\alpha_k, \mathbf{w}),$$

and define

$$X'' \stackrel{\text{def.}}{=} \overline{p''^{-1}(p(X \cap \mathfrak{M}_{k; \varepsilon_k(X)}(\mathbf{v}, \mathbf{w})))}.$$

(Note that  $Q_k(\mathbf{v} - \varepsilon_k(X)\alpha_k, \mathbf{w})$  is a vector bundle of rank  $\text{wt}_k(X) + 2\varepsilon_k(X) = \varepsilon_k(X) + \varphi_k(X)$ . So the Grassmann bundle  $p''$  is nonempty by the assumption  $\varphi_k(X) > 0$ .) We define an operator  $\tilde{f}_k: \text{Irr } \tilde{\mathfrak{Z}} \rightarrow \text{Irr } \tilde{\mathfrak{Z}} \sqcup \{0\}$  by

$$(4.2) \quad \tilde{f}_k(X) \stackrel{\text{def.}}{=} \begin{cases} X'' & \text{if } \varphi_k(X) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then the following is clear.

**Proposition 4.3.**  *$\text{Irr } \tilde{\mathfrak{Z}}$  together with  $\text{wt}$ ,  $\varepsilon_k$ ,  $\varphi_k$ ,  $\tilde{e}_k$ ,  $\tilde{f}_k$  is a normal crystal.*

**Lemma 4.4.**  *$\text{Irr } \tilde{\mathfrak{Z}}$  is generated by the subset  $\{X \mid \varepsilon_k(X) = 0 \text{ for all } k \in I\}$  as a crystal.*

*Proof.* If  $\varepsilon_{k_1}(X) > 0$  for some  $k_1$ , then we have  $X = \tilde{f}_{k_1} \tilde{e}_{k_1}(X)$ . If  $\varepsilon_{k_2}(\tilde{e}_{k_1}(X)) > 0$ , then  $X = \tilde{f}_{k_2} \tilde{f}_{k_1}(\tilde{e}_{k_2} \tilde{e}_{k_1} X)$ . We continue this procedure successively. Since the total sum  $\sum_k \dim V_k$  of dimensions decreases under this, it will eventually stop at  $\tilde{e}_{k_N} \cdots \tilde{e}_{k_1} X$  with  $\varepsilon_k(\tilde{e}_{k_N} \cdots \tilde{e}_{k_1} X) = 0$  for all  $k \in I$ .  $\square$

Let  $\text{Irr } \mathfrak{L}(\mathbf{w})$  be the set of irreducible components of  $\mathfrak{L}(\mathbf{w})$ . It is a disjoint union of the set of irreducible components of  $\mathfrak{L}(\mathbf{v}, \mathbf{w})$  for various  $\mathbf{v}$ . Since  $\mathfrak{L}(\mathbf{v}, \mathbf{w}) \subset \tilde{\mathfrak{Z}} \cap \mathfrak{M}(\mathbf{v}, \mathbf{w})$  and both have dimension  $= \frac{1}{2} \dim \mathfrak{M}(\mathbf{v}, \mathbf{w})$ , we have an inclusion

$$\text{Irr } \mathfrak{L}(\mathbf{w}) \subset \text{Irr } \tilde{\mathfrak{Z}}.$$

We restrict the maps  $\text{wt}$ ,  $\varepsilon_k$ ,  $\varphi_k$  to  $\text{Irr } \mathfrak{L}(\mathbf{w})$ . Moreover, it is clear that if  $X \in \text{Irr } \mathfrak{L}(\mathbf{w})$  and  $\tilde{e}_k(X) \neq 0$  (resp.  $\tilde{f}_k(X) \neq 0$ ), then  $\tilde{e}_k(X) \in \text{Irr } \mathfrak{L}(\mathbf{w})$  (resp.  $\tilde{f}_k(X) \in \text{Irr } \mathfrak{L}(\mathbf{w})$ ). Thus  $\text{Irr } \mathfrak{L}(\mathbf{w})$  inherits the structure of the crystal from that of  $\text{Irr } \tilde{\mathfrak{Z}}$ . In fact, this crystal structure on  $\text{Irr } \mathfrak{L}(\mathbf{w})$  was introduced in [30], and our definition here is a straightforward modification. The crystal structure in [loc. cit.] was motivated by a similar construction by Lusztig [19].

Note that  $\mathfrak{M}(0, \mathbf{w})$  consists of a single point. So  $\mathfrak{L}(0, \mathbf{w}) = \tilde{\mathfrak{Z}}(\mathbf{w}^1; \mathbf{w}^2) \cap \mathfrak{M}(0, \mathbf{w}) = \mathfrak{M}(0, \mathbf{w})$ . Let  $[0]_{\mathbf{w}}$  denote this point considered as an irreducible component of  $\mathfrak{L}(\mathbf{w})$ .

**Proposition 4.5.**  *$\text{Irr } \mathfrak{L}(\mathbf{w})$  is a highest weight normal crystal, which is the strictly embedded crystal of  $\text{Irr } \tilde{\mathfrak{Z}}$  generated by  $[0]_{\mathbf{w}}$ .*

*Proof.* By Lemma 4.4  $\text{Irr } \mathfrak{L}(\mathbf{w})$  is generated by  $\{X \in \text{Irr } \mathfrak{L}(\mathbf{w}) \mid \varepsilon_k(X) = 0 \text{ for all } k\}$ . But this set consists of the single element  $[0]_{\mathbf{w}}$  as shown in the proof of [30, 7.2].  $\square$

By Proposition 3.14, we have a natural bijection between sets

$$\text{Irr } \mathfrak{L}(\mathbf{w}^1) \times \text{Irr } \mathfrak{L}(\mathbf{w}^2) \cong \text{Irr } \tilde{\mathfrak{Z}}.$$

Hereafter these two sets are identified and their element is denoted by  $X^1 \otimes X^2$  for  $X^1 \in \text{Irr } \mathfrak{L}(\mathbf{w}^1)$ ,  $X^2 \in \text{Irr } \mathfrak{L}(\mathbf{w}^2)$ . We have

$$\text{wt}(X^1 \otimes X^2) = \mathbf{w} - \mathbf{v} = \mathbf{w}^1 - \mathbf{v}^1 + \mathbf{w}^2 - \mathbf{v}^2 = \text{wt}(X^1) + \text{wt}(X^2),$$

if  $X^1$  (resp.  $X^2$ ) is an irreducible component of  $\mathfrak{L}(\mathbf{v}^1, \mathbf{w}^1)$  (resp.  $\mathfrak{L}(\mathbf{v}^2, \mathbf{w}^2)$ ). Thus we have (1.9a).

The following is one of the main results in this article.

**Theorem 4.6.** *The crystal  $\text{Irr } \tilde{\mathfrak{Z}}$  is isomorphic to  $\text{Irr } \mathfrak{L}(\mathbf{w}^1) \otimes \text{Irr } \mathfrak{L}(\mathbf{w}^2)$  as a crystal.*

Together with Proposition 4.5, this theorem implies that  $\{\text{Irr } \mathfrak{L}(\mathbf{w}) \mid \mathbf{w} \in P^+\}$  is a closed family of highest weight normal crystals. By Proposition 1.14 we have

**Corollary 4.7.**  *$\text{Irr } \mathfrak{L}(\mathbf{w})$  is isomorphic to  $\mathcal{B}(\mathbf{w})$  as a crystal for any  $\mathbf{w} \in P^+$ .*

The proof of Theorem 4.6 occupies the rest of this section. A completely different proof of Corollary 4.7 will be given in §8.

Let  $X^1 \otimes X^2 \in \text{Irr } \tilde{\mathfrak{Z}}$ . Take a generic element  $[B, i, j]$  in  $X^1 \otimes X^2$ . By Proposition 3.13 there exists a direct sum decomposition  $V = V^1 \oplus V^2$  such that

- (a)  $V^2$  is invariant under  $B$  and satisfies  $j(W^2) \subset V^2$ ,  $i(V^2) \subset W^2$ .
- (b) The data obtained by the restriction of  $[B, i, j]$  to  $V^2$ ,  $W^2$  (well-defined thanks to (a)) is contained in  $X^2$ .
- (c) The data induced on  $V^1 = V/V^2$ ,  $W^1 = W/W^2$  from  $[B, i, j]$  (well-defined thanks to (a)) is contained in  $X^1$ .

In particular, the homomorphisms  $\sigma_k, \tau_k$  decompose into the following three parts:

$$\begin{aligned} \sigma_k^1: V_k^1 &\rightarrow \bigoplus_{h:\text{in}(h)=k} V_{\text{out}(h)}^1 \oplus W_k^1, & \tau_k^1: \bigoplus_{h:\text{in}(h)=k} V_{\text{out}(h)}^1 \oplus W_k^1 &\rightarrow V_k^1, \\ \sigma_k^{21}: V_k^1 &\rightarrow \bigoplus_{h:\text{in}(h)=k} V_{\text{out}(h)}^2 \oplus W_k^2, & \tau_k^{21}: \bigoplus_{h:\text{in}(h)=k} V_{\text{out}(h)}^1 \oplus W_k^1 &\rightarrow V_k^2, \\ \sigma_k^2: V_k^2 &\rightarrow \bigoplus_{h:\text{in}(h)=k} V_{\text{out}(h)}^2 \oplus W_k^2, & \tau_k^2: \bigoplus_{h:\text{in}(h)=k} V_{\text{out}(h)}^2 \oplus W_k^2 &\rightarrow V_k^2. \end{aligned}$$

The equation  $\tau_k \sigma_k = 0$  is equivalent to  $\tau_k^1 \sigma_k^1 = 0$ ,  $\tau_k^2 \sigma_k^2 = 0$ ,  $\tau_k^{21} \sigma_k^1 + \tau_k^2 \sigma_k^{21} = 0$ . In particular,  $\tau_k^{21}$  induces a homomorphism  $\bigoplus_{h:\text{in}(h)=k} V_{\text{out}(h)}^1 \oplus W_k^1 / \text{Im } \sigma_k^1 \rightarrow V_k^2 / \text{Im } \tau_k$ . Let  $\overline{\tau_k^{21}}: \text{Ker } \tau_k^1 / \text{Im } \sigma_k^1 \rightarrow V_k^2 / \text{Im } \tau_k^2$  be its restriction.

The inclusion  $V_k^2 \subset V_k^1 \oplus V_k^2$  induces a homomorphism

$$\text{Ker } \tau_k^2 / \text{Im } \sigma_k^2 \rightarrow \text{Ker } \tau_k / \text{Im } \sigma_k.$$

The projection  $V_k^1 \oplus V_k^2 \rightarrow V_k^1$  induces a homomorphism

$$\text{Ker } \tau_k / \text{Im } \sigma_k \rightarrow \text{Ker } \overline{\tau_k^{21}}.$$

Combining these, we have a three-term sequence of vector bundles:

$$(4.8) \quad 0 \rightarrow \text{Ker } \tau_k^2 / \text{Im } \sigma_k^2 \rightarrow \text{Ker } \tau_k / \text{Im } \sigma_k \rightarrow \text{Ker } \overline{\tau_k^{21}} \rightarrow 0.$$

**Lemma 4.9.** (4.8) is an exact sequence. (This holds any  $[B, i, j]$ , not necessarily generic.)

*Proof.* By the definition, it is clear that the composite of homomorphisms is 0. The exactness at the middle and the right terms are clear. We now prove the exactness at the left term.

Suppose that  $x \in \text{Ker } \tau_k^2$ , considered as an element of  $\left(\bigoplus_{h:\text{in}(h)=k} V_{\text{out}(h)}^1 \oplus W_k^2\right) \oplus \text{Ker } \tau_k^2$ , is equal to  $\sigma_k(y)$  for some  $y = y^1 \oplus y^2 \in V_k^1 \oplus V_k^2$ . Since we have  $\sigma_k^1(y^1) = 0$ , the injectivity of  $\sigma_k^1$  implies that  $y^1 = 0$ . Then  $\sigma_k(y) = x$  implies that  $\sigma_k(y^2) = x$ . Thus we get the exactness.  $\square$

**Lemma 4.10.** Suppose  $\varepsilon_k(X^1 \otimes X^2) = 0$ . Then

- (1)  $\varepsilon_k(X^1) = 0$ ,  $\dim \text{Ker } \overline{\tau_k^{21}} = \text{wt}_k(X^1) - \varepsilon_k(X^2)$ . In particular,  $\text{wt}_k(X^1) \geq \varepsilon_k(X^2)$ .
- (2) We have

$$\tilde{f}_k^r(X^1 \otimes X^2) = \begin{cases} \tilde{f}_k^r X^1 \otimes X^2 & \text{if } r \leq \text{wt}_k(X^1) - \varepsilon_k(X^2), \\ \tilde{f}_k^{\text{wt}_k(X^1) - \varepsilon_k(X^2)} X^1 \otimes \tilde{f}_k^{r - \text{wt}_k(X^1) + \varepsilon_k(X^2)} X^2 & \text{otherwise.} \end{cases}$$

*Proof.* (1) By the assumption,  $\tau_k$  is surjective. It is true if and only if the following two statements hold:

(a)  $\tau_k^1$  is surjective,

(b)  $\text{Ker } \tau_k^1 \oplus \bigoplus_{h:\text{in}(h)=k} V_{\text{out}(h)}^2 \oplus W_k^2 \xrightarrow{[\tau_k^{21}|_{\text{Ker } \tau_k^1} \quad \tau_k^2]} V_k^2$  is surjective.

Then the second statement is equivalent to

(b') The homomorphism  $\overline{\tau_k^{21}}$  is surjective.

By (a) we have  $\varepsilon_k(X^1) = 0$ . Therefore  $\dim \text{Ker } \tau_k^1 / \text{Im } \sigma_k^1 = \varphi_k(X^1) = \text{wt}_k(X^1)$ . By (b') we have  $\dim \text{Ker } \overline{\tau_k^{21}} = \text{wt}_k(X^1) - \varepsilon_k(X^2)$ .

(2) A subspace  $S$  of  $\text{Ker } \tau_k / \text{Im } \sigma_k$  defines subspaces  $S^1$  of  $\text{Ker } \overline{\tau_k^{21}}$  and  $S^2$  of  $\text{Ker } \tau_k^2 / \text{Im } \sigma_k^2$  with an exact sequence  $0 \rightarrow S^2 \rightarrow S \rightarrow S^1 \rightarrow 0$ . For a generic  $S$ , we have

$$\begin{aligned} \dim S^1 &= \min(\dim S, \dim \text{Ker } \overline{\tau_k^{21}}), \\ \dim S^2 &= \max(0, \dim S - \dim \text{Ker } \overline{\tau_k^{21}}). \end{aligned}$$

Therefore we have the assertion.  $\square$

Now Theorem 4.6 follows from the following elementary lemma.

**Lemma 4.11.** *Let  $\mathcal{B}_1, \mathcal{B}_2$  be normal crystals. Suppose that  $\mathcal{B}_1 \times \mathcal{B}_2$  has a structure of a normal crystal such that*

- (a) (1.9a) holds for any  $b_1 \otimes b_2$ .
- (b) If  $\varepsilon_k(b_1 \otimes b_2) = 0$ , then we have (1.9b), (1.9d) and

$$\tilde{f}_k^r(b_1 \otimes b_2) = \begin{cases} \tilde{f}_k^r b_1 \otimes b_2 & \text{if } r \leq \text{wt}_k(b_1) - \varepsilon_k(b_2), \\ \tilde{f}_k^{\text{wt}_k(b_1) - \varepsilon_k(b_2)} b_1 \otimes \tilde{f}_k^{r - \text{wt}_k(b_1) + \varepsilon_k(b_2)} b_2 & \text{otherwise.} \end{cases}$$

(We denote  $(b_1, b_2)$  by  $b_1 \otimes b_2$ .) Then the crystal structure coincides with that of the tensor product.

*Proof.* Take  $b_1 \otimes b_2 \in \mathcal{B}_1 \times \mathcal{B}_2$  with  $\varepsilon_k(b_1 \otimes b_2) = r$ . Then we have  $b_1 \otimes b_2 = \tilde{f}_k^r(b'_1 \otimes b'_2)$  for some  $b'_1 \otimes b'_2 \in \mathcal{B}_1 \times \mathcal{B}_2$ . We have  $\varepsilon_k(b'_1 \otimes b'_2) = 0$ . Hence the condition (b) implies

$$\begin{aligned} b_1 &= \begin{cases} \tilde{f}_k^r b'_1 & \text{if } r \leq \text{wt}_k(b'_1) - \varepsilon_k(b'_2), \\ \tilde{f}_k^{\text{wt}_k(b'_1) - \varepsilon_k(b'_2)} b'_1 & \text{otherwise,} \end{cases} \\ b_2 &= \begin{cases} b'_2 & \text{if } r \leq \text{wt}_k(b'_1) - \varepsilon_k(b'_2), \\ \tilde{f}_k^{r - \text{wt}_k(b'_1) + \varepsilon_k(b'_2)} b'_2 & \text{otherwise.} \end{cases} \end{aligned}$$

First consider the case  $r < \text{wt}_k(b'_1) - \varepsilon_k(b'_2)$ . Then we have

$$\begin{aligned} (4.12) \quad \tilde{e}_k(b_1 \otimes b_2) &= \tilde{f}_k^{r-1}(b'_1 \otimes b'_2) = \tilde{f}_k^{r-1} b'_1 \otimes b'_2 = \tilde{e}_k b_1 \otimes b_2, \\ \tilde{f}_k(b_1 \otimes b_2) &= \tilde{f}_k^{r+1}(b'_1 \otimes b'_2) = \tilde{f}_k^{r+1} b'_1 \otimes b'_2 = \tilde{f}_k b_1 \otimes b_2. \end{aligned}$$

The first two inequalities give us

$$(4.13) \quad \varepsilon_k(b_2) - \text{wt}_k(b_1) = \varepsilon_k(b'_2) - \text{wt}_k(b'_1) + 2r < r = \varepsilon_k(b_1),$$

where we have used the assumption in the inequality. Thus we have checked (1.9b), (1.9d), (1.9e) in this case.

Next consider the case  $r = \text{wt}_k(b'_1) - \varepsilon_k(b'_2)$ . We have (4.12) except that the last line is replaced by

$$\tilde{f}_k(b_1 \otimes b_2) = \tilde{f}_k^{r+1}(b'_1 \otimes b'_2) = \tilde{f}_k^r b'_1 \otimes \tilde{f}_k b'_2 = b_1 \otimes \tilde{f}_k b_2.$$

The inequality (4.13) is replaced by

$$\varepsilon_k(b_2) - \text{wt}_k(b_1) = \varepsilon_k(b_1).$$

We also have (1.9b), (1.9d), (1.9e) in this case.

Finally consider the case  $r > \text{wt}_k(b'_1) - \varepsilon_k(b'_2)$ . Then we have

$$\begin{aligned}\varepsilon_k(b_1) &= \text{wt}_k(b'_1) - \varepsilon_k(b'_2), & \varepsilon_k(b_2) &= r - \text{wt}_k(b'_1) + 2\varepsilon_k(b'_2), \\ \tilde{e}_k(b_1 \otimes b_2) &= \tilde{f}_k^{r-1}(b'_1 \otimes b'_2) = b_1 \otimes \tilde{e}_k b_2, \\ \tilde{f}_k(b_1 \otimes b_2) &= \tilde{f}_k^{r+1}(b'_1 \otimes b'_2) = b_1 \otimes \tilde{f}_k b_2.\end{aligned}$$

We have

$$\begin{aligned}\varepsilon_k(b_2) - \text{wt}_k(b_1) &= r - \text{wt}_k(b'_1) + 2\varepsilon_k(b'_2) - \text{wt}_k(b'_1) + 2(\text{wt}_k(b'_1) - \varepsilon_k(b'_2)) \\ &= r > \text{wt}_k(b'_1) - \varepsilon_k(b'_2) = \varepsilon_k(b_1),\end{aligned}$$

where we have used the assumption in the inequality. Thus we have (1.9b),(1.9d),(1.9e) in this case.

We have checked (1.9b),(1.9d),(1.9e) in all cases, and (1.9c) follows from  $\varphi_k(b_1 \otimes b_2) = \varepsilon_k(b_1 \otimes b_2) + \text{wt}_k(b_1 \otimes b_2)$ .  $\square$

*Remark 4.14.* The rule (1.9b) is equivalent to that  $\overline{\tau_k^{21}}$  is full rank, i.e.,

$$\text{rank } \overline{\tau_k^{21}} = \max(\dim(\text{Ker } \tau_k^1 / \text{Im } \sigma_k^1), \dim(V_k^2 / \text{Im } \tau_k^2)).$$

*Remark 4.15.* Saito [35] proved Corollary 4.7 using the main result of [16]. On the other hand, one can show the main result of [16] from Corollary 4.7. The detail is left for the reader.

## 5. $\mathfrak{g}$ -MODULE STRUCTURE

Let  $Z(\mathbf{w}) \stackrel{\text{def.}}{=} \mathfrak{M}(\mathbf{w}) \times_{\mathfrak{M}_0(\infty, \mathbf{w})} \mathfrak{M}(\mathbf{w})$ . Let  $H_{\text{top}}(Z(\mathbf{w}), \mathbb{Q})$  be the top degree part of the Borel-Moore homology group of  $Z(\mathbf{w})$ . More precisely, it is the subspace

$$\prod'_{\mathbf{v}, \mathbf{v}'} H_{\dim_{\mathbb{C}} \mathfrak{M}(\mathbf{v}', \mathbf{w}) \times \mathfrak{M}(\mathbf{v}, \mathbf{w})}(Z(\mathbf{w}) \cap \mathfrak{M}(\mathbf{v}', \mathbf{w}) \times \mathfrak{M}(\mathbf{v}, \mathbf{w}), \mathbb{Q}),$$

of the direct products consisting elements  $(F_{\mathbf{v}, \mathbf{v}'})$  such that

- (1) for fixed  $\mathbf{v}$ ,  $F_{\mathbf{v}, \mathbf{v}'} = 0$  for all but finitely many choices of  $\mathbf{v}'$ ,
- (2) for fixed  $\mathbf{v}'$ ,  $F_{\mathbf{v}, \mathbf{v}'} = 0$  for all but finitely many choices of  $\mathbf{v}$ .

By the convolution product (see [32, §8]), it is an associative algebra with  $1 = \sum_{\mathbf{v}} [\Delta(\mathbf{v}, \mathbf{w})]$ . Let  $\omega: \mathfrak{M}(\mathbf{v}', \mathbf{w}) \times \mathfrak{M}(\mathbf{v}, \mathbf{w}) \rightarrow \mathfrak{M}(\mathbf{v}, \mathbf{w}) \times \mathfrak{M}(\mathbf{v}', \mathbf{w})$  be the flip of the components. By a main result of [30, §9], the assignment

$$\begin{aligned}P^* \ni h &\longmapsto \sum_{\mathbf{v}} \langle h, \mathbf{w} - \mathbf{v} \rangle [\Delta(\mathbf{v}, \mathbf{w})] \\ e_k &\longmapsto \sum_{\mathbf{v}} [\mathfrak{P}_k(\mathbf{v}, \mathbf{w})], & f_k &\longmapsto \sum_{\mathbf{v}} \pm [\omega(\mathfrak{P}_k(\mathbf{v}', \mathbf{w}))]\end{aligned}$$

defines an algebra homomorphism

$$\mathbf{U}(\mathfrak{g}) \rightarrow H_{\text{top}}(Z(\mathbf{w}), \mathbb{Q}).$$

Here the sign  $\pm$  can be explicitly given by  $\mathbf{v}, \mathbf{w}$ . But its explicit form is not important for our purpose. (In [30] the direct sum was used, and  $\mathbf{U}(\mathfrak{g})$  was replaced by the modified enveloping algebra.)

Let  $H_{\text{top}}(\mathfrak{L}(\mathbf{w}), \mathbb{Q})$ ,  $H_{\text{top}}(\tilde{\mathfrak{Z}}, \mathbb{Q})$  be the top degree part of the Borel-Moore homology group of  $\mathfrak{L}(\mathbf{w})$ ,  $\tilde{\mathfrak{Z}}$ , where ‘the top degree part’ means, as above, the complex dimension of  $\mathfrak{M}(\mathbf{v}, \mathbf{w})$ .



(The sum is the usual direct sum.) The fundamental classes of irreducible components of  $\mathfrak{L}(\mathbf{w})$  (resp.  $\tilde{\mathfrak{Z}}$ ) give a basis of  $H_{\text{top}}(\mathfrak{L}(\mathbf{w}), \mathbb{Q})$  (resp.  $H_{\text{top}}(\tilde{\mathfrak{Z}}, \mathbb{Q})$ ). An irreducible component and its fundamental class is identified hereafter. Recall that the inclusion  $\mathfrak{L}(\mathbf{w}) \subset \tilde{\mathfrak{Z}}$  induces an inclusion  $\text{Irr } \mathfrak{L}(\mathbf{w}) \subset \text{Irr } \tilde{\mathfrak{Z}}$ . Hence we have an injection

$$(5.1) \quad H_{\text{top}}(\mathfrak{L}(\mathbf{w}), \mathbb{Q}) \rightarrow H_{\text{top}}(\tilde{\mathfrak{Z}}, \mathbb{Q}).$$

Since  $\mathfrak{L}(\mathbf{w})$  and  $\tilde{\mathfrak{Z}}$  are  $\pi$ -saturated (Lemma 3.5), the convolution makes  $H_{\text{top}}(\mathfrak{L}(\mathbf{w}), \mathbb{Q})$  and  $H_{\text{top}}(\tilde{\mathfrak{Z}}, \mathbb{Q})$  into  $H_{\text{top}}(Z(\mathbf{w}), \mathbb{Q})$ -modules. Moreover, (5.1) is a morphism of  $H_{\text{top}}(Z(\mathbf{w}), \mathbb{Q})$ -modules. Hence they can be considered as  $\mathfrak{g}$ -modules. By [30, 10.2],  $H_{\text{top}}(\mathfrak{L}(\mathbf{w}), \mathbb{Q})$  is the simple  $\mathfrak{g}$ -module with highest weight  $\mathbf{w}$ . The highest weight vector is the fundamental class  $[0]_{\mathbf{w}}$  of  $\mathfrak{L}(0, \mathbf{w}) = \mathfrak{M}(0, \mathbf{w}) = \text{point}$ .

**Theorem 5.2.**  *$H_{\text{top}}(\tilde{\mathfrak{Z}}, \mathbb{Q})$  is isomorphic to  $V(\mathbf{w}^1) \otimes V(\mathbf{w}^2)$  as a  $\mathfrak{g}$ -module.*

*Proof.* By the argument in [30, 9.3], the module  $H_{\text{top}}(\tilde{\mathfrak{Z}}, \mathbb{Q})$  is integrable. Since the category of integrable  $\mathfrak{g}$ -modules is completely reducible, it is enough to show that characters of  $H_{\text{top}}(\tilde{\mathfrak{Z}}, \mathbb{Q})$ , and  $V(\mathbf{w}^1) \otimes V(\mathbf{w}^2)$  are equal. By the definition of  $h$ , we have

$$hX = \langle h, \text{wt } X \rangle X \quad \text{for } X \in \text{Irr } \tilde{\mathfrak{Z}}.$$

Namely the weight of  $X$  as crystal is the same as the weight defined by the  $\mathfrak{g}$ -module structure. Thus Theorem 4.6 and Corollary 4.7 imply the equality of the character.

In fact, we do not need the full power of Theorem 4.6. We only need two things: (1) there exists an isomorphism of sets  $\text{Irr } \tilde{\mathfrak{Z}} \cong \text{Irr } \mathfrak{L}(\mathbf{w}^1) \times \text{Irr } \mathfrak{L}(\mathbf{w}^2)$  satisfying  $\text{wt}(X^1 \otimes X^2) = \text{wt}(X^1) + \text{wt}(X^2)$ , and (2)  $H_{\text{top}}(\mathfrak{L}(\mathbf{w}^p), \mathbb{C})$  is isomorphic to  $V(\mathbf{w}^p)$  ( $p = 1, 2$ ). Thus our proof follows from Proposition 3.14(1) and [30].  $\square$

This theorem is abstract. It is desirable to have a concrete construction of the isomorphism. For example, the isomorphism can be made so that the injective homomorphism (5.1) is identified with a homomorphism  $V(\mathbf{w}^1 + \mathbf{w}^2) \rightarrow V(\mathbf{w}^1) \otimes V(\mathbf{w}^2)$ , sending  $b_{\mathbf{w}^1 + \mathbf{w}^2}$  to  $b_{\mathbf{w}^1} \otimes b_{\mathbf{w}^2}$ , where  $b_{\lambda}$  is the highest weight vector of  $V(\lambda)$ . However this condition does not characterize the isomorphism. In the rest of this section, we study  $H_{\text{top}}(\tilde{\mathfrak{Z}}, \mathbb{Q})$  further for this desire.

Recall that  $\tilde{\mathfrak{Z}}(0, \mathbf{w}^1; \mathbf{v}, \mathbf{w}^2)$  is a (possibly empty) closed subvariety of  $\tilde{\mathfrak{Z}} \cap \mathfrak{M}(\mathbf{v}, \mathbf{w})$  (Remark 3.8). By Proposition 3.13, it is a vector bundle over  $\mathfrak{L}(0, \mathbf{w}^1) \times \mathfrak{L}(\mathbf{v}, \mathbf{w}^2) \cong \mathfrak{L}(\mathbf{v}, \mathbf{w}^2)$ . Let  $\tilde{\mathfrak{Z}}_1(\mathbf{v}) \stackrel{\text{def.}}{=} \tilde{\mathfrak{Z}}(0, \mathbf{w}^1; \mathbf{v}, \mathbf{w}^2)$ ,  $\mathfrak{Z}_1(\mathbf{v}) \stackrel{\text{def.}}{=} \mathfrak{Z}(0, \mathbf{w}^1; \mathbf{v}, \mathbf{w}^2)$ ,  $\mathfrak{Z}_1 \stackrel{\text{def.}}{=} \bigsqcup_{\mathbf{v}} \mathfrak{Z}_1(\mathbf{v})$ , and  $\tilde{\mathfrak{Z}}_1 \stackrel{\text{def.}}{=} \bigsqcup_{\mathbf{v}} \tilde{\mathfrak{Z}}_1(\mathbf{v})$ .

**Lemma 5.3.** *Let  $\hat{\mathfrak{P}}_k^{(n)}(\mathbf{v}, \mathbf{w})$  denote the intersection of  $\mathfrak{P}_k^{(n)}(\mathbf{v}, \mathbf{w})$  and  $\mathfrak{M}(\mathbf{v} - n\alpha_k, \mathbf{w}) \times \mathfrak{Z}_1(\mathbf{v})$  (as submanifolds of  $\mathfrak{M}(\mathbf{v} - n\alpha_k, \mathbf{w}) \times \mathfrak{M}(\mathbf{v}, \mathbf{w})$ ). Let  $p$  (resp.  $p'$ ) denote the projection  $\mathfrak{Z}_1(\mathbf{v}) \rightarrow \mathfrak{M}(\mathbf{v}, \mathbf{w})$  (resp.  $\mathfrak{Z}_1(\mathbf{v} - n\alpha_k) \rightarrow \mathfrak{M}(\mathbf{v} - n\alpha_k, \mathbf{w})$ ).*

(1)  $\hat{\mathfrak{P}}_k^{(n)}(\mathbf{v}, \mathbf{w})$  is contained in  $\mathfrak{Z}_1(\mathbf{v} - n\alpha_k) \times \mathfrak{Z}_1(\mathbf{v})$

(2) *The restriction of  $\text{id} \times p: \mathfrak{Z}_1(\mathbf{v} - n\alpha_k) \times \mathfrak{Z}_1(\mathbf{v}) \rightarrow \mathfrak{Z}_1(\mathbf{v} - n\alpha_k) \times \mathfrak{M}(\mathbf{v}, \mathbf{w}^2)$  to  $\hat{\mathfrak{P}}_k^{(n)}(\mathbf{v}, \mathbf{w})$  gives an isomorphism*

$$\hat{\mathfrak{P}}_k^{(n)}(\mathbf{v}, \mathbf{w}) \xrightarrow[\cong]{\text{id} \times p} (p' \times \text{id})^{-1}(\mathfrak{P}_k^{(n)}(\mathbf{v}, \mathbf{w}^2)) \cong L(W^1, V')|_{\mathfrak{P}_k^{(n)}(\mathbf{v}, \mathbf{w}^2)},$$

where  $\mathfrak{P}_k^{(n)}(\mathbf{v}, \mathbf{w}^2)$  is the Hecke correspondence in  $\mathfrak{M}(\mathbf{v} - n\alpha_k, \mathbf{w}^2) \times \mathfrak{M}(\mathbf{v}, \mathbf{w}^2)$ , and  $V'$  is the pull-back of the tautological vector bundle of the first factor of  $\mathfrak{M}(\mathbf{v} - n\alpha_k, \mathbf{w}) \times \mathfrak{M}(\mathbf{v}, \mathbf{w})$ .

(3) *The intersection is transverse.*

*Proof.* By Proposition 3.13,  $\mathfrak{Z}_1(\mathbf{v})$  is the total space of the vector bundle  $L(W^1, V)$  over  $\mathfrak{M}(0, \mathbf{w}^1) \times \mathfrak{M}(\mathbf{v}, \mathbf{w}^2)$  (use  $V^1 = 0$ ,  $V^2 = V$ ). Then

$$\widehat{\mathfrak{P}}_k^{(n)}(\mathbf{v}, \mathbf{w}) \cong L(W^1, V')|_{\mathfrak{P}_k^{(n)}(\mathbf{v}, \mathbf{w}^2)},$$

where the right hand side is the total space of the restriction of the vector bundle  $L(W^1, V')$ . Since  $\mathfrak{Z}_1(\mathbf{v} - n\alpha_k)$  is the total space of  $L(W^1, V')$ , the statements (1) and (2) are clear.

Let us describe the tangent bundles of submanifolds on the intersection. From now, restrictions or pull-backs of vector bundles are denoted by the same notation as original bundles. The tangent bundle of  $\mathfrak{M}(\mathbf{v} - n\alpha_k, \mathbf{w}) \times \mathfrak{M}(\mathbf{v}, \mathbf{w})$  appears in an exact sequence

$$\begin{aligned} 0 \rightarrow L(V', W^1) \oplus L(W^1, V') \oplus L(V, W^1) \oplus L(W^1, V) \\ \rightarrow T\mathfrak{M}(\mathbf{v} - n\alpha_k, \mathbf{w}) \oplus T\mathfrak{M}(\mathbf{v}, \mathbf{w}) \rightarrow T\mathfrak{M}(\mathbf{v} - n\alpha_k, \mathbf{w}^2) \oplus T\mathfrak{M}(\mathbf{v}, \mathbf{w}^2) \rightarrow 0. \end{aligned}$$

The tangent bundle of  $\mathfrak{M}(\mathbf{v} - n\alpha_k, \mathbf{w}) \times \mathfrak{Z}_1(\mathbf{v})$  appears as

$$\begin{aligned} 0 \rightarrow L(V', W^1) \oplus L(W^1, V') \oplus L(W^1, V) \\ \rightarrow T\mathfrak{M}(\mathbf{v} - n\alpha_k, \mathbf{w}) \oplus T\mathfrak{Z}_1(\mathbf{v}) \rightarrow T\mathfrak{M}(\mathbf{v} - n\alpha_k, \mathbf{w}^2) \oplus T\mathfrak{M}(\mathbf{v}, \mathbf{w}^2) \rightarrow 0. \end{aligned}$$

The tangent bundle of  $\mathfrak{P}_k^{(n)}(\mathbf{v}, \mathbf{w})$  appears as

$$0 \rightarrow L(W^1, V') \oplus L(V, W^1) \rightarrow T\mathfrak{P}_k^{(n)}(\mathbf{v}, \mathbf{w}) \rightarrow T\mathfrak{P}_k^{(n)}(\mathbf{v}, \mathbf{w}^2) \rightarrow 0.$$

Now the transversality is clear. □

Since  $\widetilde{\mathfrak{Z}}_1$  is a vector bundle over  $\mathfrak{L}(\mathbf{w}^2)$ , we have the Thom isomorphism

$$(5.4) \quad H_{\text{top}}(\mathfrak{L}(\mathbf{w}^2), \mathbb{Q}) \cong H_{\text{top}}(\widetilde{\mathfrak{Z}}_1, \mathbb{Q}).$$

Under this isomorphism  $\text{Irr } \mathfrak{L}(\mathbf{w}^2) \ni X$  is mapped to  $[0]_{\mathbf{w}^1} \otimes X \in \text{Irr } \widetilde{\mathfrak{Z}}_1$ .

**Proposition 5.5.** (1) *As a  $\mathfrak{g}$ -module,  $H_{\text{top}}(\widetilde{\mathfrak{Z}}, \mathbb{Q})$  is generated by its subspace  $H_{\text{top}}(\widetilde{\mathfrak{Z}}_1, \mathbb{Q})$ .*

(2) *The subspace  $H_{\text{top}}(\widetilde{\mathfrak{Z}}_1, \mathbb{Q})$  is invariant under the action of  $\mathbf{U}(\mathfrak{g})^+$ , and the Thom isomorphism (5.4) is compatible with the  $\mathbf{U}(\mathfrak{g})^+$ -module structures.*

*Proof.* (1) The proof of [30, 10.2] shows that  $H_{\text{top}}(\widetilde{\mathfrak{Z}}, \mathbb{Q})$  is generated by elements  $X^1 \otimes X^2 \in \text{Irr } \widetilde{\mathfrak{Z}}$  with  $\varepsilon_k(X^1 \otimes X^2) = 0$  for all  $k \in I$ . By Lemma 4.10(1), we have  $\varepsilon_k(X^1) = 0$  for all  $k \in I$ . As we already used in the proof of Proposition 4.5, this implies that  $X^1$  is the highest weight vector  $[0]_{\mathbf{w}^1}$ . Hence  $X^1 \otimes X^2$  is contained in  $H_{\text{top}}(\widetilde{\mathfrak{Z}}_1, \mathbb{Q})$ .

(2) The first statement follows from Lemma 5.3(1), i.e.,  $\mathfrak{P}_k(\mathbf{w}) \cap (\mathfrak{M}(\mathbf{w}) \times \mathfrak{Z}_1) = \widehat{\mathfrak{P}}_k(\mathbf{w}) \subset \mathfrak{Z}_1 \times \mathfrak{Z}_1$ . The rest of proof is based on the argument in [32, §8]. Let  $i: \mathfrak{Z}_1 \rightarrow \mathfrak{M}(\mathbf{w})$  be the inclusion. By the pull-back with support map with respect to  $\text{id} \times i: \mathfrak{M}(\mathbf{w}) \times \mathfrak{Z}_1 \rightarrow \mathfrak{M}(\mathbf{w}) \times \mathfrak{M}(\mathbf{w})$  ([32, 6.4]), we have

$$H_{\text{top}}(\widehat{\mathfrak{P}}_k(\mathbf{w}), \mathbb{Q}) \xrightarrow{(\text{id} \times i)^*} H_{\text{top}}(\widehat{\mathfrak{P}}_k(\mathbf{w}), \mathbb{Q}).$$

By [32, 8.2.3] this map is compatible with the convolution product, that is, the following is a commutative diagram:

$$\begin{array}{ccc} H_{\text{top}}(\mathfrak{P}_k(\mathbf{w}), \mathbb{Q}) \otimes H_{\text{top}}(\widetilde{\mathfrak{Z}}_1, \mathbb{Q}) & \longrightarrow & H_{\text{top}}(\widetilde{\mathfrak{Z}}_1, \mathbb{Q}) \\ (\text{id} \times i)^* \otimes \text{id} \downarrow & & \parallel \\ H_{\text{top}}(\widehat{\mathfrak{P}}_k(\mathbf{w}), \mathbb{Q}) \otimes H_{\text{top}}(\widetilde{\mathfrak{Z}}_1, \mathbb{Q}) & \longrightarrow & H_{\text{top}}(\widetilde{\mathfrak{Z}}_1, \mathbb{Q}), \end{array}$$

where the upper horizontal arrow is the convolution relative to  $\mathfrak{M}(\mathbf{w})$ , and the lower horizontal arrow is the convolution relative to  $\mathfrak{Z}_1$ . Furthermore, Lemma 5.3(3) implies that  $(\text{id} \times i)^*[\mathfrak{P}_k(\mathbf{w})] = [\widehat{\mathfrak{P}}_k(\mathbf{w})]$ .

By Lemma 5.3(2), we have an isomorphism

$$H_{\text{top}}(\widehat{\mathfrak{P}}_k(\mathbf{w}), \mathbb{Q}) \xrightarrow[\cong]{((p' \times \text{id})^*)^{-1}(\text{id} \times p)_*} H_{\text{top}}(\mathfrak{P}_k(\mathbf{w}^2), \mathbb{Q}),$$

where  $(p' \times \text{id})^*$  is the Thom isomorphism. By [32, 8.3.5], it is compatible with the convolution product, that is, the following is commutative:

$$\begin{array}{ccc} H_{\text{top}}(\widehat{\mathfrak{P}}_k(\mathbf{w}), \mathbb{Q}) \otimes H_{\text{top}}(\widetilde{\mathfrak{Z}}_1, \mathbb{Q}) & \longrightarrow & H_{\text{top}}(\widetilde{\mathfrak{Z}}_1, \mathbb{Q}) \\ ((p' \times \text{id})^*)^{-1}(\text{id} \times p)_* \otimes (p^*)^{-1} \downarrow & & \downarrow (p^*)^{-1} \\ H_{\text{top}}(\mathfrak{P}_k(\mathbf{w}^2), \mathbb{Q}) \otimes H_{\text{top}}(\mathfrak{L}(\mathbf{w}^2), \mathbb{Q}) & \longrightarrow & H_{\text{top}}(\mathfrak{L}(\mathbf{w}^2), \mathbb{Q}), \end{array}$$

where the horizontal arrows are the convolution product relative to  $\mathfrak{Z}_1$  and  $\mathfrak{M}(\mathbf{w}^2)$ , and  $p^*$  is the Thom isomorphism (5.4). Moreover we have  $((p' \times \text{id})^*)^{-1}(\text{id} \times p)_*[\widehat{\mathfrak{P}}_k(\mathbf{w})] = [\mathfrak{P}_k(\mathbf{w}^2)]$  by Lemma 5.3(2). Combining two commutative diagrams, we get the assertion.  $\square$

**Conjecture 5.6.** There exists a unique  $\mathfrak{g}$ -module isomorphism

$$H_{\text{top}}(\mathfrak{L}(\mathbf{w}^1), \mathbb{Q}) \otimes H_{\text{top}}(\mathfrak{L}(\mathbf{w}^2), \mathbb{Q}) \rightarrow H_{\text{top}}(\widetilde{\mathfrak{Z}}, \mathbb{Q})$$

such that its restriction to  $[0]_{\mathbf{w}^1} \otimes H_{\text{top}}(\mathfrak{L}(\mathbf{w}^2), \mathbb{Q})$  is (5.4), composed with the inclusion  $H_{\text{top}}(\widetilde{\mathfrak{Z}}_1, \mathbb{Q}) \rightarrow H_{\text{top}}(\widetilde{\mathfrak{Z}}, \mathbb{Q})$ . (The uniqueness is clear from Proposition 5.5(1).)

The rest of this section is devoted to the proof of the conjecture when  $\mathfrak{g}$  is of type *ADE*. When  $\mathfrak{g}$  is of type *ADE*,  $V(\mathbf{w}^2)$  contains the lowest weight vector  $q_{\mathbf{m}^2}$ , where  $\mathbf{m}^2$  is the weight of  $q_{\mathbf{m}^2}$ . We denote by  $_{\mathbf{m}^2}[q]$  the corresponding element in  $\text{Irr } \mathfrak{L}(\mathbf{w}^2) \subset H_{\text{top}}(\mathfrak{L}(\mathbf{w}^2), \mathbb{Q})$ . For a later purpose, we denote by  $[0]_{\mathbf{w}^1} \circ_{\mathbf{m}^2}[q]$  the irreducible component of  $\widetilde{\mathfrak{Z}}$ , corresponding to the tensor product of  $[0]_{\mathbf{w}^1}$  and  $_{\mathbf{m}^2}[q]$  under the identification  $\text{Irr } \widetilde{\mathfrak{Z}} = \text{Irr } \mathfrak{L}(\mathbf{w}^1) \otimes \text{Irr } \mathfrak{L}(\mathbf{w}^2)$ .

**Lemma 5.7.** For each  $k \in I$ , we have

$$(5.8a) \quad e_k^{-\langle h_k, \mathbf{m}^2 \rangle + 1}([0]_{\mathbf{w}^1} \circ_{\mathbf{m}^2}[q]) = 0,$$

$$(5.8b) \quad f_k^{\langle h_k, \mathbf{w}^1 \rangle + 1}([0]_{\mathbf{w}^1} \circ_{\mathbf{m}^2}[q]) = 0,$$

$$(5.8c) \quad h([0]_{\mathbf{w}^1} \circ_{\mathbf{m}^2}[q]) = \langle h, \mathbf{w}^1 + \mathbf{m}^2 \rangle [0]_{\mathbf{w}^1} \circ_{\mathbf{m}^2}[q].$$

*Proof.* The first equation follows from Proposition 5.5(2) and  $e_k^{-\langle h_k, \mathbf{m}^2 \rangle + 1} _{\mathbf{m}^2}[q] = 0$  (a well-known property of the lowest weight vector). The last equation follows from Theorem 4.6 and the compatibility of weight structures for the crystal and the  $\mathfrak{g}$ -module.

Let us show (5.8b). Assume that the left hand side is nonzero. Then  $\mathfrak{P}_k^{(\langle h_k, \mathbf{w}^1 \rangle + 1)}(\mathbf{w})$  intersects with  $p_2^{-1}([0]_{\mathbf{w}^1} \circ_{\mathbf{m}^2}[q])$ , where  $p_2: \mathfrak{M}(\mathbf{w}) \times \mathfrak{M}(\mathbf{w}) \rightarrow \mathfrak{M}(\mathbf{w})$  is the second projection. Take a point  $[B, i, j]$  in the image under  $p_2$  of a point in the intersection and consider the exact sequence (4.8). Then  $p_2^{-1}([B, i, j]) \cap \mathfrak{P}_k^{(\langle h_k, \mathbf{w}^1 \rangle + 1)}(\mathbf{w})$  is identified with the Grassmann variety of subspaces  $S$  in  $\text{Ker } \tau_k / \text{Im } \sigma_k$  with  $\dim S = \langle h_k, \mathbf{w}^1 \rangle + 1$  by [30, 4.5] or [32, 5.4.3].

Since  $\mathbf{m}^2$  is the lowest weight, we have  $\widetilde{f}_k _{\mathbf{m}^2}[q] = 0$ . Therefore,

$$0 = \varphi_k(_{\mathbf{m}^2}[q]) = \dim(\text{Ker } \tau_k^2 / \text{Im } \sigma_k^2).$$

We must be careful with the second equality since it holds only for a *generic*  $[B, i, j]$  in general. However,  $\mathfrak{M}(\mathbf{w}^2 - \mathbf{m}^2, \mathbf{w}^2) = \mathfrak{L}(\mathbf{w}^2 - \mathbf{m}^2, \mathbf{w}^2) = \mathbf{m}^2[q]$  is isomorphic to  $\mathfrak{M}(0, \mathbf{w}^2)$ , which is a single point, as seen by the Weyl group symmetry [33]. So the equality holds for our  $[B, i, j]$ . Thus we have  $\text{Ker } \tau_k / \text{Im } \sigma_k \cong \text{Ker } \overline{\tau_k^{21}}$  by the exactness of (4.8). But

$$\dim \text{Ker } \overline{\tau_k^{21}} \leq \dim (\text{Ker } \tau_k^1 / \text{Im } \sigma_k^1) = \dim W_k^1 = \langle h_k, \mathbf{w}^1 \rangle.$$

So the Grassmann variety is empty. This contradiction comes from our assumption. Hence we have (5.8b).  $\square$

**Theorem 5.9.** *There exists a unique  $\mathfrak{g}$ -module isomorphism  $H_{\text{top}}(\mathfrak{L}(\mathbf{w}^1), \mathbb{Q}) \otimes H_{\text{top}}(\mathfrak{L}(\mathbf{w}^2), \mathbb{Q}) \rightarrow H_{\text{top}}(\tilde{\mathfrak{Z}}, \mathbb{Q})$  sending  $[0]_{\mathbf{w}^1} \otimes \mathbf{m}^2[q]$  to  $[0]_{\mathbf{w}^1 \circ \mathbf{m}^2}[q]$ . Moreover, its restriction to  $[0]_{\mathbf{w}^1} \otimes H_{\text{top}}(\mathfrak{L}(\mathbf{w}^2), \mathbb{Q})$  is (5.4), composed with the inclusion  $H_{\text{top}}(\tilde{\mathfrak{Z}}_1, \mathbb{Q}) \rightarrow H_{\text{top}}(\tilde{\mathfrak{Z}}, \mathbb{Q})$ .*

*Proof.* We identify  $H_{\text{top}}(\mathfrak{L}(\mathbf{w}^1), \mathbb{Q}) \otimes H_{\text{top}}(\mathfrak{L}(\mathbf{w}^2), \mathbb{Q})$  with the tensor product  $V(\mathbf{w}^1) \otimes V(\mathbf{w}^2)$ .

By [20, 23.3.6] the assignment  $\mathbf{U}(\mathfrak{g}) \ni u \mapsto u([0]_{\mathbf{w}^1} \otimes \mathbf{m}^2[q])$  is a surjective homomorphism  $\mathbf{U}(\mathfrak{g}) \rightarrow H_{\text{top}}(\mathfrak{L}(\mathbf{w}^1), \mathbb{Q}) \otimes H_{\text{top}}(\mathfrak{L}(\mathbf{w}^2), \mathbb{Q})$  with kernel

$$\sum_k \mathbf{U}(\mathfrak{g}) f_k^{w_k^1+1} + \sum_k \mathbf{U}(\mathfrak{g}) e_k^{-m_k^2+1}.$$

Therefore Lemma 5.7 implies that  $\mathbf{U}(\mathfrak{g}) \ni u \mapsto u([0]_{\mathbf{w}^1} \circ \mathbf{m}^2[q])$  factors through a  $\mathfrak{g}$ -module homomorphism  $H_{\text{top}}(\mathfrak{L}(\mathbf{w}^1), \mathbb{Q}) \otimes H_{\text{top}}(\mathfrak{L}(\mathbf{w}^2), \mathbb{Q}) \rightarrow H_{\text{top}}(\tilde{\mathfrak{Z}}, \mathbb{Q})$  sending  $[0]_{\mathbf{w}^1} \otimes \mathbf{m}^2[q]$  to  $[0]_{\mathbf{w}^1 \circ \mathbf{m}^2}[q]$ . The uniqueness is clear.

Since  $H_{\text{top}}(\mathfrak{L}(\mathbf{w}^2), \mathbb{Q}) = \mathbf{U}(\mathfrak{g})^+ q_{\mathbf{m}^2}$ , Proposition 5.5(2) implies

$$H_{\text{top}}(\tilde{\mathfrak{Z}}_1, \mathbb{Q}) = \mathbf{U}(\mathfrak{g})^+ ([0]_{\mathbf{w}^1} \circ \mathbf{m}^2[q]).$$

Together with Proposition 5.5(1), we have the surjectivity of the homomorphism.

Now we compare the dimensions of the domain and the target. We have

$$\begin{aligned} \dim H_{\text{top}}(\mathfrak{L}(\mathbf{w}^1), \mathbb{Q}) \otimes H_{\text{top}}(\mathfrak{L}(\mathbf{w}^2), \mathbb{Q}) &= \# \text{Irr } \mathfrak{L}(\mathbf{w}^1) \# \text{Irr } \mathfrak{L}(\mathbf{w}^2) \\ &= \# \text{Irr } \tilde{\mathfrak{Z}} = \dim H_{\text{top}}(\tilde{\mathfrak{Z}}, \mathbb{Q}). \end{aligned}$$

Thus the surjective homomorphism must be an isomorphism.

The second statement follows from Proposition 5.5(2).  $\square$

## 6. $\mathbf{U}_q(\mathbf{Lg})$ -MODULE STRUCTURE

In this section, we assume  $\mathfrak{g}$  is of type *ADE*.

For  $p = 1, 2$ , let  $H_{W^p}$  be a maximal torus of  $G_{W^p}$ . Let  $H = H_{W^1} \times H_{W^2}$ ,  $\tilde{H}_{W^p} = H_{H^p} \times \mathbb{C}^*$ ,  $\tilde{H} = H \times \mathbb{C}^*$ . The representation ring  $R(\mathbb{C}^*)$  of  $\mathbb{C}^*$  is  $\mathbb{Z}[q, q^{-1}]$ , where  $q$  is the class of the canonical 1-dimensional representation of  $\mathbb{C}^*$ . Considering  $H$  as a subgroup of  $G_W$ , we make  $\tilde{H}$  act on  $\mathfrak{M}(\mathbf{w})$ . The action preserves  $\mathfrak{M}(\mathbf{w}^1) \times \mathfrak{M}(\mathbf{w}^2)$ ,  $\tilde{\mathfrak{Z}}$  and  $\tilde{\mathfrak{Z}}_1$ . Let  $K^{\tilde{H}}(Z(\mathbf{w}))$  be the equivariant  $K$ -homology group of  $Z(\mathbf{w})$ . It is an associative  $R(\tilde{H}) \cong R(H)[q, q^{-1}]$ -algebra with unit under the convolution product. Let  $\Delta$  denote the diagonal embedding  $\mathfrak{M}(\mathbf{v}, \mathbf{w}) \rightarrow \mathfrak{M}(\mathbf{v}, \mathbf{w}) \times \mathfrak{M}(\mathbf{v}, \mathbf{w})$ . Let  $\iota: \mathfrak{P}_k(\mathbf{v}, \mathbf{w}) \rightarrow Z(\mathbf{w}) \cap \mathfrak{M}(\mathbf{v}', \mathbf{w}) \times \mathfrak{M}(\mathbf{v}, \mathbf{w})$  denote the

inclusion. By [32], the assignment

$$\begin{aligned} q^h &\longmapsto \sum_{\mathbf{v}} q^{\langle h, \mathbf{w} - \mathbf{v} \rangle} \Delta_* \mathcal{O}_{\mathfrak{M}(\mathbf{v}, \mathbf{w})}, \\ \psi_k^\pm(z) &\longmapsto \sum_{\mathbf{v}} q^{\text{rank } C_k^\bullet(\mathbf{v}, \mathbf{w})} \Delta_* \left( \frac{\bigwedge_{-1/qz} (C_k^\bullet(\mathbf{v}, \mathbf{w}))}{\bigwedge_{-q/z} (C_k^\bullet(\mathbf{v}, \mathbf{w}))} \right)^\pm, \\ e_{k,r} &\longmapsto \sum_{\mathbf{v}} \pm \iota_* (q^{-1} V/V')^{\otimes r-s} \otimes \mathcal{L}, \\ f_{k,r} &\longmapsto \sum_{\mathbf{v}} \pm' \omega_* \iota_* (q^{-1} V/V')^{\otimes r-s'} \otimes \mathcal{L}' \end{aligned}$$

defines an algebra homomorphism

$$\mathbf{U}_q^{\mathbb{Z}}(\mathbf{Lg}) \otimes_{\mathbb{Z}[q, q^{-1}]} R(\tilde{H}) \rightarrow K^{\tilde{H}}(Z(\mathbf{w}))/\text{torsion}.$$

Here  $\pm, \pm' \in \{1, -1\}$ ,  $s, s' \in \mathbb{Z}$  can be given in terms of  $\mathbf{v}, \mathbf{w}, k$  as [32, 9.3.2], but their explicit forms are not important for our later discussion. Similarly  $\mathcal{L}, \mathcal{L}'$  are line bundles whose explicit forms are not important. Those terms are independent of  $r$ .

The convolution make  $K^{\tilde{H}}(\mathfrak{L}(\mathbf{w}))$ ,  $K^{\tilde{H}}(\tilde{\mathfrak{Z}})$ ,  $K^{\tilde{H}}(\mathfrak{Z})$ ,  $K^{\tilde{H}}(\mathfrak{M}(\mathbf{w}))$  into  $K^{\tilde{H}}(Z(\mathbf{w}))/\text{torsion}$ -modules. (Note all these are free.) Moreover, the inclusions in Theorem 3.9(3) respect the module structure. Let  $[0]_{\mathbf{w}}$  be the class represented by the structure sheaf of  $\mathfrak{M}(0, \mathbf{w}) = \mathfrak{L}(0, \mathbf{w}) = \text{point}$ . By [32, 12.3.2, 13.3.1],  $[0]_{\mathbf{w}}$  is an  $l$ -highest weight vector with Drinfeld polynomial  $P_k(u) = \bigwedge_{-u} q^{-1} W_k$ , that is,

$$\begin{aligned} e_{k,r}[0]_{\mathbf{w}} &= 0 \quad \text{for any } k \in I, r \in \mathbb{Z}, \\ K^{\tilde{H}}(\mathfrak{L}(\mathbf{w})) &= \left( \mathbf{U}_q^{\mathbb{Z}}(\mathbf{Lg})^- \otimes_{\mathbb{Z}[q, q^{-1}]} R(\tilde{H}) \right) [0]_{\mathbf{w}}, \\ \psi_k^\pm(z)[0]_{\mathbf{w}} &= \left( \bigwedge_{-1/qz} (q^{-1} - q) W_k \right)^\pm [0]_{\mathbf{w}}. \end{aligned}$$

This  $K^{\tilde{H}}(\mathfrak{L}(\mathbf{w}))$ , more precisely, its Weyl group invariant part  $K^{G_w \times \mathbb{C}^*}(\mathfrak{L}(\mathbf{w}))$ , is the universal standard module  $M(\mathbf{w})$  mentioned in the introduction and §1.3.

Let us take a closed subvariety  $\tilde{\mathfrak{Z}}_1$  be as in §5. We have the Thom isomorphism in the  $K$ -theory:

$$(6.1) \quad K^{\tilde{H}}(\mathfrak{L}(\mathbf{w}^2)) \cong K^{\tilde{H}}(\tilde{\mathfrak{Z}}_1),$$

where  $H_{W^1}$  acts trivially on  $\mathfrak{L}(\mathbf{w}^2)$ . By Theorem 3.9(2) the inclusion  $\tilde{\mathfrak{Z}}_1 \subset \tilde{\mathfrak{Z}}$  induces an injective  $R(\tilde{H})$ -homomorphism

$$(6.2) \quad K^{\tilde{H}}(\tilde{\mathfrak{Z}}_1) \hookrightarrow K^{\tilde{H}}(\tilde{\mathfrak{Z}}).$$

The following can be proved exactly as in Proposition 5.5. (For (1), we use [32, 12.3.2] instead of [30, 10.2]. And for (2), we must use  $\mathfrak{P}_k^{(n)}(\mathbf{w})$  corresponding to divided powers.)

**Proposition 6.3.** (1) As a  $\mathbf{U}_q^{\mathbb{Z}}(\mathbf{Lg}) \otimes_{\mathbb{Z}[q, q^{-1}]} R(\tilde{H})$ -module,  $K^{\tilde{H}}(\tilde{\mathfrak{Z}})$  is generated by  $K^{\tilde{H}}(\tilde{\mathfrak{Z}}_1)$ .

(2)  $K^{\tilde{H}}(\tilde{\mathfrak{Z}}_1)$  is invariant under the action of  $\mathbf{U}_q^{\mathbb{Z}}(\mathbf{Lg})^+$ , and the Thom isomorphism (6.1) is compatible with the  $\mathbf{U}_q^{\mathbb{Z}}(\mathbf{Lg})^+$ -module structures (up to shift  $e_{k,r} \rightarrow e_{k,r+s}$  and invertible elements in  $R(\tilde{H})$ ).

Let  $\mathfrak{R}(\tilde{H})$  be the fraction field of the  $R(\tilde{H})$ . If  $M$  is an  $R(\tilde{H})$ -module,  $M \otimes_{R(\tilde{H})} \mathfrak{R}(\tilde{H})$  is denoted by  $M_{\mathfrak{R}}$ . By the localization theorem [36], we have

$$K^{\tilde{H}}(*)_{\mathfrak{R}} \xrightarrow[\cong]{i^*} K^{\tilde{H}}(*^{\tilde{H}})_{\mathfrak{R}},$$

where  $*$  =  $Z(\mathbf{w})$ ,  $\mathfrak{L}(\mathbf{w})$ ,  $\tilde{\mathfrak{Z}}$ ,  $\mathfrak{Z}$ , or  $\mathfrak{M}(\mathbf{w})$ ,  $*^{\tilde{H}}$  denotes the fixed point set, and  $i$  denotes the inclusion  $\mathfrak{M}(\mathbf{w})^{\tilde{H}} \times \mathfrak{M}(\mathbf{w})^{\tilde{H}} \rightarrow \mathfrak{M}(\mathbf{w})$  (if  $*$  =  $Z(\mathbf{w})$ ) or  $\mathfrak{M}(\mathbf{w})^{\tilde{H}} \rightarrow \mathfrak{M}(\mathbf{w})$  (if  $*$   $\neq$   $Z(\mathbf{w})$ ). For  $Z(\mathbf{w})$ , we replace  $i^*$  by  $r = 1 \boxtimes (\bigwedge_{-1} N^*)^{-1} i^*$ , where  $N$  is the normal bundle of  $\mathfrak{M}(\mathbf{w})^{\tilde{H}}$  in  $\mathfrak{M}(\mathbf{w})$  as in [4, 5.11]. Then it is compatible with the convolution product, i.e.,  $r$  is an algebra homomorphism, and the following is commutative:

$$\begin{array}{ccc} K^{\tilde{H}}(Z(\mathbf{w}))_{\mathfrak{R}} \otimes_{\mathfrak{R}(\tilde{H})} K^{\tilde{H}}(*)_{\mathfrak{R}} & \longrightarrow & K^{\tilde{H}}(*)_{\mathfrak{R}} \\ r \otimes i^* \downarrow \cong & & \cong \downarrow i^* \\ K^{\tilde{H}}(Z(\mathbf{w})^{\tilde{H}})_{\mathfrak{R}} \otimes_{\mathfrak{R}(\tilde{H})} K^{\tilde{H}}(*^{\tilde{H}})_{\mathfrak{R}} & \longrightarrow & K^{\tilde{H}}(*^{\tilde{H}})_{\mathfrak{R}}, \end{array}$$

where  $*$  =  $\mathfrak{L}(\mathbf{w})$ ,  $\tilde{\mathfrak{Z}}$ ,  $\mathfrak{Z}$ , or  $\mathfrak{M}(\mathbf{w})$ , and the horizontal arrows are convolution relative to  $\mathfrak{M}(\mathbf{w})$  and  $\mathfrak{M}(\mathbf{w})^{\tilde{H}}$  respectively. Therefore  $K^{\tilde{H}}(*^{\tilde{H}})_{\mathfrak{R}}$  has a structure of a  $\mathbf{U}_q(\mathbf{Lg}) \otimes_{\mathbb{Q}(q)} \mathfrak{R}(\tilde{H})$ -module.

On the other hand,  $K^{\tilde{H}}(\mathfrak{L}(\mathbf{w}^1))_{\mathfrak{R}} \otimes K^{\tilde{H}}(\mathfrak{L}(\mathbf{w}^2))_{\mathfrak{R}}$  can be considered as a  $\mathbf{U}_q(\mathbf{Lg}) \otimes_{\mathbb{Q}(q)} \mathfrak{R}(\tilde{H})$ -module by the comultiplication (1.2). The following lemma first appeared in [32, 14.1.2].

**Lemma 6.4.** *There exists a unique  $\mathbf{U}_q(\mathbf{Lg}) \otimes_{\mathbb{Q}(q)} \mathfrak{R}(\tilde{H})$ -module isomorphism*

$$\Phi_{\mathfrak{R}}: K^{\tilde{H}}(\mathfrak{L}(\mathbf{w}^1))_{\mathfrak{R}} \otimes_{\mathfrak{R}(\tilde{H})} K^{\tilde{H}}(\mathfrak{L}(\mathbf{w}^2))_{\mathfrak{R}} \rightarrow K^{\tilde{H}}(\tilde{\mathfrak{Z}})_{\mathfrak{R}},$$

sending  $[0]_{\mathbf{w}^1} \otimes [0]_{\mathbf{w}^2}$  to  $[0]_{\mathbf{w}}$ .

*Proof.* We have (cf. [32, 4.2.2])

$$(6.5) \quad \mathfrak{L}(\mathbf{w})^{\tilde{H}} = \tilde{\mathfrak{Z}}^{\tilde{H}} = \mathfrak{Z}^{\tilde{H}} = \mathfrak{M}(\mathbf{w})^{\tilde{H}} = \mathfrak{M}(\mathbf{w}^1)^{\tilde{H}} \times \mathfrak{M}(\mathbf{w}^2)^{\tilde{H}} = \mathfrak{L}(\mathbf{w}^1)^{\tilde{H}} \times \mathfrak{L}(\mathbf{w}^2)^{\tilde{H}}.$$

By the argument in [32, 14.1.2], this implies that  $K^{\tilde{H}}(\tilde{\mathfrak{Z}}^{\tilde{H}})_{\mathfrak{R}} = K^{\tilde{H}}(\tilde{\mathfrak{Z}})_{\mathfrak{R}}$  is a simple  $\mathbf{U}_q(\mathbf{Lg}) \otimes_{\mathbb{Q}(q)} \mathfrak{R}(\tilde{H})$ -module. Its  $k$ th Drinfeld polynomial is given by

$$\bigwedge_{-u} q^{-1} W_k = \bigwedge_{-u} q^{-1} W_k^1 \otimes \bigwedge_{-u} q^{-1} W_k^2.$$

(Here Drinfeld polynomials have values in  $R(\tilde{H})$ .) It is equal to the product of the Drinfeld polynomials of  $K^{\tilde{H}}(\mathfrak{L}(\mathbf{w}^1))_{\mathfrak{R}}$  and  $K^{\tilde{H}}(\mathfrak{L}(\mathbf{w}^2))_{\mathfrak{R}}$ . Therefore, it is a subquotient of the tensor product module

$$K^{\tilde{H}}(\mathfrak{L}(\mathbf{w}^1))_{\mathfrak{R}} \otimes_{\mathfrak{R}(\tilde{H})} K^{\tilde{H}}(\mathfrak{L}(\mathbf{w}^2))_{\mathfrak{R}}.$$

(A well-known argument based on Lemma 1.20(1).) By the localization theorem, (6.5) and the Künneth isomorphism (Theorem 3.4), we know that the dimensions of the both hand sides are equal. (See also the remark below.) Hence we have the unique  $\mathbf{U}_q(\mathbf{Lg}) \otimes_{\mathbb{Q}(q)} \mathfrak{R}(\tilde{H})$ -isomorphism  $\Phi_{\mathfrak{R}}$  sending  $[0]_{\mathbf{w}^1} \otimes [0]_{\mathbf{w}^2}$  to  $[0]_{\mathbf{w}}$ .  $\square$

*Remark 6.6.* The left hand side of (6.9) is isomorphic to  $K^{\tilde{H}}(\mathfrak{L}(\mathbf{w}^1)^{\tilde{H}})_{\mathfrak{R}} \otimes_{\mathfrak{R}(\tilde{H})} K^{\tilde{H}}(\mathfrak{L}(\mathbf{w}^2)^{\tilde{H}})_{\mathfrak{R}}$  by the localization theorem. Combining it with (6.5) and the Künneth isomorphism, we have an isomorphism (of  $\mathfrak{R}(\tilde{H})$ -modules) between the left-hand side and the right-hand side. But it does not respect  $\mathbf{U}_q(\mathbf{Lg})$ -module structures. In fact, a computation in [37, 7.4] implies that

the isomorphism respects  $\mathbf{U}_q(\mathbf{Lg})$ -module structures, if we endow the left hand side with a  $\mathbf{U}_q(\mathbf{Lg})$ -module structure given by Drinfeld's new comultiplication (after an explicit twist). Thus our map  $\Phi_{\mathfrak{R}}$  should be given explicitly by factors of the universal  $R$ -matrix as in [17]. However, it will become difficult (at least for the author) to show the commutativity of the diagram (6.9) below, in the formula in [17]. This is the reason why we do not use the explicit form of  $\Phi_{\mathfrak{R}}$ , unlike [37].

Let  $\mathcal{L}(\mathbf{w}^1)^{\tilde{H}} = \mathfrak{M}(\mathbf{w}^1)^{\tilde{H}} = \bigsqcup_{\rho} \mathfrak{M}(\rho; \mathbf{w}^1)$  and  $\mathcal{L}(\mathbf{w}^2)^{\tilde{H}} = \mathfrak{M}(\mathbf{w}^2)^{\tilde{H}} = \bigsqcup_{\rho'} \mathfrak{M}(\rho'; \mathbf{w}^2)$  be decomposition into connected components. Then we have direct sum decomposition

$$K^{\tilde{H}}(\mathcal{L}(\mathbf{w}^1))_{\mathfrak{R}} = \bigoplus_{\rho} K^{\tilde{H}}(\mathfrak{M}(\rho; \mathbf{w}^1))_{\mathfrak{R}}, \quad K^{\tilde{H}}(\mathcal{L}(\mathbf{w}^2))_{\mathfrak{R}} = \bigoplus_{\rho'} K^{\tilde{H}}(\mathfrak{M}(\rho'; \mathbf{w}^2))_{\mathfrak{R}}.$$

By [32, 13.4.5], these are  $l$ -weight space decomposition. Here  $l$ -weights are elements in  $\mathfrak{R}(\tilde{H})[[z^{\pm}]]^I$ . Under  $\mathfrak{R}(\tilde{H}) = \mathfrak{R}(\tilde{H}_{W^1}) \otimes_{\mathbb{Q}(q)} \mathfrak{R}(\tilde{H}_{W^2})$ , they have forms of  $f \otimes 1$  and  $1 \otimes g$  respectively. Therefore, by Lemma 1.20(1),

$$(6.7) \quad K^{\tilde{H}}(\mathcal{L}(\mathbf{w}^1))_{\mathfrak{R}} \otimes_{\mathfrak{R}(\tilde{H})} K^{\tilde{H}}(\mathcal{L}(\mathbf{w}^2))_{\mathfrak{R}} = \bigoplus_{\rho, \rho'} K^{\tilde{H}}(\mathfrak{M}(\rho; \mathbf{w}^1))_{\mathfrak{R}} \otimes_{\mathfrak{R}(\tilde{H})} K^{\tilde{H}}(\mathfrak{M}(\rho'; \mathbf{w}^2))_{\mathfrak{R}}$$

is the  $l$ -weight space decomposition. The  $l$ -weight is the form of  $f \otimes g$ . Therefore each summand has distinct  $l$ -weights.

**Lemma 6.8.** *The following diagram is commutative:*

$$(6.9) \quad \begin{array}{ccc} K^{\tilde{H}}(\mathcal{L}(\mathbf{w}^2)) & \xrightarrow{(6.1)} & K^{\tilde{H}}(\tilde{\mathfrak{Z}}_1) \\ \downarrow \heartsuit & & \downarrow (6.2) \\ K^{\tilde{H}}(\mathcal{L}(\mathbf{w}^1)) \otimes_{R(\tilde{H})} K^{\tilde{H}}(\mathcal{L}(\mathbf{w}^2)) & & K^{\tilde{H}}(\tilde{\mathfrak{Z}}) \\ \otimes_{\mathfrak{R}(\tilde{H})} \downarrow & & \downarrow \otimes_{\mathfrak{R}(\tilde{H})} \\ K^{\tilde{H}}(\mathcal{L}(\mathbf{w}^1))_{\mathfrak{R}} \otimes_{\mathfrak{R}(\tilde{H})} K^{\tilde{H}}(\mathcal{L}(\mathbf{w}^2))_{\mathfrak{R}} & \xrightarrow[\Phi_{\mathfrak{R}}]{\cong} & K^{\tilde{H}}(\tilde{\mathfrak{Z}})_{\mathfrak{R}}, \end{array}$$

where  $\heartsuit$  is the inclusion

$$K^{\tilde{H}}(\mathcal{L}(\mathbf{w}^2)) \ni E \mapsto [0]_{\mathbf{w}^1} \otimes E \in K^{\tilde{H}}(\mathcal{L}(\mathbf{w}^1)) \otimes_{R(\tilde{H})} K^{\tilde{H}}(\mathcal{L}(\mathbf{w}^2)).$$

*Proof.* Let  $\mathbf{m}^2[q]$  be the class represented by the structure sheaf of  $\mathfrak{M}(\mathbf{w}^2 - \mathbf{m}^2, \mathbf{w}^2) = \mathcal{L}(\mathbf{w}^2 - \mathbf{m}^2, \mathbf{w}^2) = \text{point}$ , as in the previous section (the lowest weight vector). We have  $f_{k,r} \mathbf{m}^2[q] = 0$  for any  $k \in I, r \in \mathbb{Z}$ . Consider the element  $T_{w_0}$  of the Braid group corresponding to the longest element  $w_0$  of the Weyl group (of  $\mathfrak{g}$ ). It acts on  $K^{\tilde{H}}(\mathcal{L}(\mathbf{w}^2))$  by [20, Part VI]. Since  $w_0 \mathbf{w} = \mathbf{m}^2$ ,  $T_{w_0}$  maps  $[0]_{\mathbf{w}^2}$  to  $\alpha \mathbf{m}^2[q]$ , where  $\alpha$  is an invertible element in  $R(\tilde{H})$ . We have

$$\left( \mathbf{U}_q^{\mathbb{Z}}(\mathbf{Lg}) \otimes_{\mathbb{Z}[q, q^{-1}]} R(\tilde{H}) \right)_{\mathbf{m}^2[q]} = \left( \mathbf{U}_q^{\mathbb{Z}}(\mathbf{Lg})^+ \otimes_{\mathbb{Z}[q, q^{-1}]} R(\tilde{H}) \right)_{\mathbf{m}^2[q]} = K^{\tilde{H}}(\mathcal{L}(\mathbf{w}^2)).$$

Combining this with Proposition 6.3, we understand that it is enough to check the commutativity of the diagram for the element  $\mathbf{m}^2[q]$ . Let  $'([0]_{\mathbf{w}^1} \otimes \mathbf{m}^2[q])$  be the image of  $\mathbf{m}^2[q]$  under the composition of (6.2) and (6.1). We want to show

$$(6.10) \quad \Phi_{\mathfrak{R}}([0]_{\mathbf{w}^1} \otimes \mathbf{m}^2[q]) = '([0]_{\mathbf{w}^1} \otimes \mathbf{m}^2[q]).$$

In the  $l$ -weight space decomposition (6.7), both hand sides of (6.10) is contained in the summand

$$K^{\tilde{H}}(\mathfrak{M}(0, \mathbf{w}^1))_{\mathfrak{R}} \otimes_{\mathfrak{R}(\tilde{H})} K^{\tilde{H}}(\mathfrak{M}(\mathbf{w}^2 - \mathbf{m}^2, \mathbf{w}^2))_{\mathfrak{R}}.$$

Recall both  $\mathfrak{M}(0, \mathbf{w}^1)$  and  $\mathfrak{M}(\mathbf{w}^2 - \mathbf{m}^2, \mathbf{w}^2)$  are a single point. Therefore the  $l$ -weight space is 1-dimensional. Thus (6.10) holds up to a nonzero constant in  $\mathfrak{R}(\tilde{H})$ .

By [20, 39.1.2], we have

$$[0]_{\mathbf{w}^2} = \alpha T_{w_0} \mathbf{m}^2[q] = \alpha e_{k_1,0}^{(a_1)} e_{k_2,0}^{(a_2)} \cdots e_{k_N,0}^{(a_N)} \mathbf{m}^2[q],$$

where  $s_{k_1} s_{k_2} \cdots s_{k_N}$  is a reduced expression of  $w_0$ , and  $a_i \in \mathbb{Z}_{\geq 0}$  is an explicitly computable natural number. By Lemma 1.20(2) we have

$$[0]_{\mathbf{w}^1} \otimes [0]_{\mathbf{w}^2} = \alpha e_{k_1,0}^{(a_1)} e_{k_2,0}^{(a_2)} \cdots e_{k_N,0}^{(a_N)} ([0]_{\mathbf{w}^1} \otimes \mathbf{m}^2[q])$$

in  $K^{\tilde{H}}(\mathfrak{L}(\mathbf{w}^1)) \otimes_{R(\tilde{H})} K^{\tilde{H}}(\mathfrak{L}(\mathbf{w}^2))$ . Therefore,

$$\alpha e_{k_1,0}^{(a_1)} e_{k_2,0}^{(a_2)} \cdots e_{k_N,0}^{(a_N)} (\Phi_{\mathfrak{R}}([0]_{\mathbf{w}^1} \otimes \mathbf{m}^2[q])) = [0]_{\mathbf{w}}.$$

On the other hand, by Proposition 6.3(2) we have

$$\alpha e_{k_1,0}^{(a_1)} e_{k_2,0}^{(a_2)} \cdots e_{k_N,0}^{(a_N)} '([0]_{\mathbf{w}^1} \otimes \mathbf{m}^2[q]) = [0]_{\mathbf{w}}.$$

We have used the commutativity of the diagram for the element  $[0]_{\mathbf{w}^2}$ , which is obvious from the definition. Therefore the constant must be 1.  $\square$

**Theorem 6.11.**  $\Phi_{\mathfrak{R}}$  induces an  $\mathbf{U}_q^{\mathbb{Z}}(\mathbf{Lg}) \otimes_{\mathbb{Z}[q,q^{-1}]} R(\tilde{H})$ -module isomorphism

$$\Phi: K^{\tilde{H}}(\mathfrak{L}(\mathbf{w}^1)) \otimes_{R(\tilde{H})} K^{\tilde{H}}(\mathfrak{L}(\mathbf{w}^2)) \xrightarrow{\cong} K^{\tilde{H}}(\tilde{\mathfrak{Z}}).$$

*Proof.* Since  $K^{\tilde{H}}(\mathfrak{L}(\mathbf{w}^1))$  is generated by  $[0]_{\mathbf{w}^1}$ ,  $K^{\tilde{H}}(\mathfrak{L}(\mathbf{w}^1)) \otimes_{R(\tilde{H})} K^{\tilde{H}}(\mathfrak{L}(\mathbf{w}^2))$  is generated by  $[0]_{\mathbf{w}^1} \otimes K^{\tilde{H}}(\mathfrak{L}(\mathbf{w}^2))$ . Therefore,  $K^{\tilde{H}}(\mathfrak{L}(\mathbf{w}^1)) \otimes_{R(\tilde{H})} K^{\tilde{H}}(\mathfrak{L}(\mathbf{w}^2))$  is mapped to  $\mathbf{U}_q^{\mathbb{Z}}(\mathbf{Lg}) K^{\tilde{H}}(\tilde{\mathfrak{Z}}_1)$  under  $\Phi_{\mathfrak{R}}$ . But it is equal to  $K^{\tilde{H}}(\tilde{\mathfrak{Z}})$  by Proposition 6.3(1).  $\square$

As an application, we have a new proof of [37, 7.12].

**Corollary 6.12.** Let  $\varepsilon \in \mathbb{C}^*$ . Suppose two  $I$ -tuple polynomials  $P^1 = (P_k^1)_{k \in I}$ ,  $P^2 = (P_k^2)_{k \in I}$  satisfy that

$$\alpha/\alpha' \notin \{\varepsilon^n \mid n \in \mathbb{Z}, n \geq 2\} \quad \text{for any pair } (\alpha, \alpha') \text{ with } P_k^1(\alpha) = 0, P_{k'}^2(\alpha') = 0 \ (k, k' \in I).$$

Then we have a unique  $\mathbf{U}_{\varepsilon}(\mathbf{Lg})$ -isomorphism

$$M(P^1 P^2) \cong M(P^1) \otimes_{\mathbb{C}} M(P^2),$$

sending  $[0]_{\mathbf{w}}$  to  $[0]_{\mathbf{w}^1} \otimes [0]_{\mathbf{w}^2}$ .

*Proof.* Take diagonal matrices  $s^1 \in H_{W^1}$ ,  $s^2 \in H_{W^2}$  whose entries are roots of  $P^1$ ,  $P^2$  respectively. The evaluation at  $a = (s^1, s^2, \varepsilon) \in H_{W^1} \times H_{W^2} \times \mathbb{C}^* = \tilde{H}$  defines a homomorphism  $R(\tilde{H}) = R(H)[q, q^{-1}] \rightarrow \mathbb{C}$ . Then we set

$$M(P^1 P^2) = K^{\tilde{H}}(\mathfrak{L}(\mathbf{w})) \otimes_{R(\tilde{H})} \mathbb{C},$$

$$M(P^1) \otimes_{\mathbb{C}} M(P^2) = K^{\tilde{H}}(\mathfrak{L}(\mathbf{w}^1)) \otimes_{R(\tilde{H})} K^{\tilde{H}}(\mathfrak{L}(\mathbf{w}^2)) \otimes_{R(\tilde{H})} \mathbb{C} = K^{\tilde{H}}(\tilde{\mathfrak{Z}}) \otimes_{R(\tilde{H})} \mathbb{C}.$$



The inclusion in Theorem 3.9(3) induces a  $\mathbf{U}_\varepsilon(\mathbf{Lg})$ -homomorphism  $M(P^1 P^2) \rightarrow M(P^1) \otimes_{\mathbb{C}} M(P^2)$ , sending  $[0]_{\mathbf{w}}$  to  $[0]_{\mathbf{w}^1} \otimes [0]_{\mathbf{w}^2}$ . If we denote by  $X_a$  the localization of an  $R(\tilde{H})$ -module  $X$  at the maximal ideal corresponding to  $a$ , the above equalities factor through as

$$M(P^1 P^2) = K^{\tilde{H}}(\mathfrak{L}(\mathbf{w}))_a \otimes_{R(\tilde{H})_a} \mathbb{C}, \quad M(P^1) \otimes M(P^2) = K^{\tilde{H}}(\tilde{\mathfrak{Z}})_a \otimes_{R(\tilde{H})_a} \mathbb{C}.$$

By the localization theorem [36], we have

$$K^{\tilde{H}}(\mathfrak{L}(\mathbf{w}))_a \cong K^{\tilde{H}}(\mathfrak{L}(\mathbf{w})^a)_a, \quad K^{\tilde{H}}(\tilde{\mathfrak{Z}})_a \cong K^{\tilde{H}}(\tilde{\mathfrak{Z}}^a)_a$$

where  $*^a$  denotes the set of points fixed by  $a$ . So the result follows from the following.

*Claim.*  $\mathfrak{L}(\mathbf{w})^a = \tilde{\mathfrak{Z}}^a$ .

Let  $W(\alpha) = W^1(\alpha) \oplus W^2(\alpha)$  be the eigenspace of  $s_1 \oplus s_2$  with eigenvalue  $\alpha$ . Let  $[B, i, j] \in \tilde{\mathfrak{Z}}^a$  and

$$f \stackrel{\text{def.}}{=} j_{\text{in}(h_N)} B_{h_N} B_{h_{N-1}} \cdots B_{h_1} i_{\text{out}(h_1)} : W_{\text{out}(h_1)} \rightarrow W_{\text{in}(h_N)},$$

where  $h_1, \dots, h_N$  is a path in our graph. Since  $[B, i, j]$  is fixed by  $a$ ,  $f$  maps  $W_{\text{out}(h_1)}(\alpha)$  to  $W_{\text{in}(h_N)}(\varepsilon^{-2-N}\alpha)$ . On the other hand, the condition  $[B, i, j] \in \tilde{\mathfrak{Z}}$  implies that  $f$  maps  $W^2$  to 0 and  $W^1$  to  $W^2$  as in the proof of Lemma 3.6. Therefore,  $f$  must be 0 by the assumption. Since the path is arbitrary, it means that  $\pi([B, i, j]) = 0$ , i.e.,  $[B, i, j] \in \mathfrak{L}(\mathbf{w})$ .  $\square$

*Remark 6.13.* In our proof of Theorem 6.11, the assumption that  $\mathbf{g}$  is of type ADE is used for the existence of the comultiplication  $\Delta$ . If one can prove the existence of it such that Lemma 1.20 still holds, then our proof goes well for a general Kac-Moody Lie algebra  $\mathbf{g}$ . Or, if one can give a different proof of the existence of the isomorphism  $\Phi$  in Theorem 6.11 as merely  $R(\tilde{H})$ -modules, without using  $\mathbf{U}_q^{\mathbb{Z}}(\mathbf{Lg})$ -module structures, then it means that one can consider  $\Phi$  as a ‘definition’ of the tensor product module.

## 7. GENERAL CASE

Almost all results in previous sections can be generalized to the case of a tensor product of more than two modules.

Let us suppose a direct sum decomposition  $W = W^1 \oplus W^2 \oplus \cdots \oplus W^N$  of  $I$ -graded vector spaces is given. Let  $H_{W^p} \subset G_{W^p}$  be the maximal torus of diagonal matrices, and let  $H \stackrel{\text{def.}}{=} H_{W^1} \times H_{W^2} \times \cdots \times H_{W^N}$ , and let  $\tilde{H} \stackrel{\text{def.}}{=} H \times \mathbb{C}^*$ . We choose a one-parameter subgroup  $\lambda : \mathbb{C}^* \rightarrow G_{W^1} \times G_{W^2} \times \cdots \times G_{W^N}$  given by

$$\lambda(t) = t^{m_1} \text{id}_{W^1} \oplus t^{m_2} \text{id}_{W^2} \oplus \cdots \oplus t^{m_N} \text{id}_{W^N},$$

with  $m_1 < m_2 < \cdots < m_N$ . Moreover we take generic  $m_i$ ’s and assume that the fixed point set  $\mathfrak{M}(\mathbf{w})^{\lambda(\mathbb{C}^*)}$  is  $\mathfrak{M}(\mathbf{w}^1) \times \mathfrak{M}(\mathbf{w}^2) \times \cdots \times \mathfrak{M}(\mathbf{w}^N)$ . We define

$$\begin{aligned} \mathfrak{Z}(\mathbf{w}^1; \mathbf{w}^2; \dots; \mathbf{w}^N) &\stackrel{\text{def.}}{=} \left\{ [B, i, j] \in \mathfrak{M}(\mathbf{w}) \mid \lim_{t \rightarrow 0} \lambda(t) * [B, i, j] \text{ exists} \right\}, \\ \tilde{\mathfrak{Z}}(\mathbf{w}^1; \mathbf{w}^2; \dots; \mathbf{w}^N) &\stackrel{\text{def.}}{=} \left\{ [B, i, j] \in \mathfrak{M}(\mathbf{w}) \mid \lim_{t \rightarrow 0} \lambda(t) * [B, i, j] \in \mathfrak{L}(\mathbf{w}^1) \times \cdots \times \mathfrak{L}(\mathbf{w}^N) \right\}. \end{aligned}$$

Equivalently, we can define inductively

$$\begin{aligned} \mathfrak{Z}(\mathbf{w}^1; \mathbf{w}^2; \dots; \mathbf{w}^N) &= \left\{ [B, i, j] \in \mathfrak{M}(\mathbf{w}) \mid \lim_{t \rightarrow 0} \lambda'(t) * [B, i, j] \in \mathfrak{Z}(\mathbf{w}^1; \mathbf{w}^2; \dots; \mathbf{w}^{N-1}) \times \mathfrak{M}(\mathbf{w}^N) \right\}, \\ \widetilde{\mathfrak{Z}}(\mathbf{w}^1; \mathbf{w}^2; \dots; \mathbf{w}^N) &= \left\{ [B, i, j] \in \mathfrak{M}(\mathbf{w}) \mid \lim_{t \rightarrow 0} \lambda'(t) * [B, i, j] \in \widetilde{\mathfrak{Z}}(\mathbf{w}^1; \mathbf{w}^2; \dots; \mathbf{w}^{N-1}) \times \mathfrak{L}(\mathbf{w}^N) \right\}, \end{aligned}$$

where

$$\lambda'(t) = \text{id}_{W^1} \oplus \dots \oplus \text{id}_{W^{N-1}} \oplus t \text{id}_{W^N}.$$

These are closed subvarieties and  $\widetilde{\mathfrak{Z}}(\mathbf{w}^1; \mathbf{w}^2; \dots; \mathbf{w}^N)$  is lagrangian. We have

- (1)  $\text{Irr } \widetilde{\mathfrak{Z}}(\mathbf{w}^1; \mathbf{w}^2; \dots; \mathbf{w}^N)$  has a structure of a crystal isomorphic to  $\mathcal{B}(\mathbf{w}^1) \otimes \dots \otimes \mathcal{B}(\mathbf{w}^N)$ . (Theorem 4.6)
- (2)  $H_{\text{top}}(\widetilde{\mathfrak{Z}}(\mathbf{w}^1; \mathbf{w}^2; \dots; \mathbf{w}^N), \mathbb{Q})$  is isomorphic to  $V(\mathbf{w}^1) \otimes \dots \otimes V(\mathbf{w}^N)$  as a  $\mathfrak{g}$ -module. (Theorem 5.2)
- (3) (When  $\mathfrak{g}$  is of type *ADE*)  $K^{\widetilde{H}}(\widetilde{\mathfrak{Z}}(\mathbf{w}^1; \mathbf{w}^2; \dots; \mathbf{w}^N))$  is isomorphic to  $K^{\widetilde{H}}(\mathfrak{L}(\mathbf{w}^1)) \otimes_{R(\widetilde{H})} \dots \otimes_{R(\widetilde{H})} K^{\widetilde{H}}(\mathfrak{L}(\mathbf{w}^N))$  as a  $\mathbf{U}_q^{\mathbb{Z}}(\mathbf{Lg}) \otimes_{\mathbb{Z}[q, q^{-1}]} R(\widetilde{H})$ -module. (Theorem 6.11)

Theorem 5.9 depends on [20, 23.3.6], which seems difficult to generalize. This is the only reason why the author does not know the generalization of Theorem 5.9.

## 8. COMBINATORIAL DESCRIPTION OF THE CRYSTAL

In this section, we give a combinatorial description of the crystal  $\text{Irr } \widetilde{\mathfrak{Z}}(\mathbf{w}^1; \dots; \mathbf{w}^N)$ .

Since  $\text{Irr } \widetilde{\mathfrak{Z}}(\mathbf{w}^1; \dots; \mathbf{w}^N) = \text{Irr } \mathfrak{L}(\mathbf{w}^1) \otimes \dots \otimes \text{Irr } \mathfrak{L}(\mathbf{w}^N)$ , it is enough to give a description of  $\text{Irr } \mathfrak{L}(\mathbf{w}^p)$ . However, we study a slightly general situation.

For a given one-parameter subgroup  $\rho_0: \mathbb{C}^* \rightarrow G_W$ , we define a  $\mathbb{C}^*$ -action on  $\mathbf{M}$  and  $\mathfrak{M}$  by

$$B_h \mapsto \begin{cases} B_h & \text{if } h \in \Omega, \\ tB_h & \text{if } h \in \overline{\Omega}, \end{cases} \quad i \mapsto \rho_0(t) * i, \quad j \mapsto \rho_0(t) * (tj).$$

We denote this  $\mathbb{C}^*$ -action by  $(B, i, j) \mapsto t \diamond (B, i, j)$  and  $[B, i, j] \mapsto t \diamond [B, i, j]$ . If  $\rho_0(t) = \text{id}_W$ , this is the  $\mathbb{C}^*$ -action considered in [28, §5]. Its crucial property is that the symplectic form  $\omega$  is transformed as  $t\omega$ . We define

$$\widetilde{\mathfrak{Z}}^\diamond \stackrel{\text{def.}}{=} \left\{ x \in \mathfrak{M}(\mathbf{w}) \mid \lim_{t \rightarrow \infty} t \diamond x \text{ exists} \right\}.$$

Note that we consider the limit for  $t \rightarrow \infty$  while we have studied the limit  $t \rightarrow 0$  in previous sections. When  $\Omega$  contains no cycle and  $\rho_0(t) = \text{id}_W$ , we have  $\widetilde{\mathfrak{Z}}^\diamond = \mathfrak{L}(\mathbf{w})$  [28, 5.3(2)].

It is easy to show that the set  $\text{Irr } \widetilde{\mathfrak{Z}}^\diamond$  of irreducible components of  $\widetilde{\mathfrak{Z}}^\diamond$  has a structure of a normal crystal as in §4.

Take a fixed point  $x$  of the  $\mathbb{C}^*$ -action and its representative  $(B, i, j)$ . Then there exists a unique homomorphism  $\rho: \mathbb{C}^* \rightarrow G_V$  such that

$$(8.1) \quad t \diamond (B, i, j) = \rho(t)^{-1} \cdot (B, i, j).$$

(The uniqueness follows from the freeness of the action of  $G_V$  on the set of stable points.) Moreover, the conjugacy class of  $\rho$  is independent of the choice of the representative  $(B, i, j)$

of  $x$ . Thus we have a decomposition

$$(8.2) \quad \mathfrak{M}(\mathbf{w})^{\mathbb{C}^*} = \bigsqcup_{\rho} \mathfrak{F}^{\diamond}(\rho),$$

where  $\mathfrak{F}^{\diamond}(\rho)$  is the set of  $[B, i, j]$  satisfying (8.1) up to conjugacy. It is clear that the conjugacy class of  $\rho$  is constant on each connected component. Furthermore, by the argument in [32, 5.5.6], we can show that each summand  $\mathfrak{F}^{\diamond}(\rho)$  is the connected component of  $\mathfrak{M}(\mathbf{w})^{\mathbb{C}^*}$  (if it is nonempty). (The  $\mathbb{C}^*$ -action used in [loc. cit.] is different from the above one. But the argument still works. Moreover the assumption  $-(\alpha_k, \alpha_l) \leq 1$  there becomes unnecessary for the above  $\mathbb{C}^*$ -action.)

Combining with the argument in [28, 5.8], we get the following.

**Proposition 8.3.** *We have a decomposition*

$$\tilde{\mathfrak{Z}}^{\diamond} = \bigsqcup_{\rho} \tilde{\mathfrak{Z}}^{\diamond}(\rho); \quad \tilde{\mathfrak{Z}}^{\diamond}(\rho) \stackrel{\text{def.}}{=} \left\{ x \in \mathfrak{M}(\mathbf{w}) \mid \lim_{t \rightarrow \infty} t \diamond x \in \mathfrak{F}^{\diamond}(\rho) \right\}.$$

The irreducible components of  $\tilde{\mathfrak{Z}}^{\diamond}$  is the closure of each summand  $\tilde{\mathfrak{Z}}^{\diamond}(\rho)$ , and they are all lagrangian subvarieties.

Here the property  $t^*\omega = t\omega$  played the crucial role.

The conjugacy class of  $\rho$  corresponds bijectively to the dimensions of its weight space in  $V_k$ . Thus we have an injective map

$$\text{Irr } \tilde{\mathfrak{Z}}^{\diamond} \rightarrow \mathbb{Z}^{I \times \mathbb{Z}}; \quad \tilde{\mathfrak{Z}}^{\diamond}(\rho) \mapsto (\dim V_k^p)_{k \in I, p \in \mathbb{Z}},$$

where  $V^p = \{v \in V \mid \rho(t) \cdot v = t^p v\}$ .

We set  $W^p = \{w \in W \mid \rho_0(t) * w = t^p w\}$ . Then (8.1) is equivalent to

$$B_h(V_{\text{out}(h)}^p) \subset \begin{cases} V_{\text{in}(h)}^p & \text{if } h \in \Omega, \\ V_{\text{in}(h)}^{p-1} & \text{if } h \in \bar{\Omega}, \end{cases} \quad i_k(W_k^p) \subset V_k^p, \quad j_k(V_k^p) \subset W_k^{p-1}.$$

By the same formula in (3.1), we define an analogous complex of vector bundles over  $\mathfrak{F}^{\diamond}(\rho)$  for each  $p, q \in \mathbb{Z}$ :

$$\text{L}(V^p, V^q) \xrightarrow{\alpha^{qp}} \begin{array}{ccc} \text{E}_{\Omega}(V^p, V^q) & & \text{L}(W^p, V^q) \\ \oplus & & \oplus \\ \text{E}_{\bar{\Omega}}(V^p, V^{q-1}) & & \text{L}(V^p, W^{q-1}) \end{array} \xrightarrow{\beta^{qp}} \text{L}(V^p, V^{q-1}),$$

where

$$\text{E}_{\Omega}(V^p, V^q) = \bigoplus_{h \in \Omega} \text{Hom}(V_{\text{out}(h)}^p, V_{\text{in}(h)}^q), \quad \text{E}_{\bar{\Omega}}(V^p, V^{q-1}) = \bigoplus_{h \in \bar{\Omega}} \text{Hom}(V_{\text{out}(h)}^p, V_{\text{in}(h)}^{q-1})$$

The restriction of the tangent bundle  $T\mathfrak{M}$  to  $\mathfrak{F}^{\diamond}(\rho)$  decomposes as  $\bigoplus_{p,q} \text{Ker } \beta^{qp} / \text{Im } \alpha^{qp}$ . Since the tangent space to the fixed point set is the 0-weight space,

$$T\mathcal{F}^{\diamond}(\rho) \cong \bigoplus_p \text{Ker } \beta^{pp} / \text{Im } \alpha^{pp}.$$

Moreover, as in Proposition 3.13 we have

$$\tilde{\mathfrak{Z}}^{\diamond}(\rho) \cong \bigoplus_{q < p} \text{Ker } \beta^{qp} / \text{Im } \alpha^{qp},$$

where the right hand side is the total space of the vector bundle. The natural map  $\tilde{\mathfrak{Z}}^\diamond(\rho) \rightarrow \mathfrak{F}^\diamond(\rho)$  is identified with the projection map of the vector bundle.

Take a point  $[B, i, j] \in \tilde{\mathfrak{Z}}^\diamond(\rho)$  and consider the complex  $C_k^\bullet$  (2.10). The homomorphisms  $\sigma_k, \tau_k$  have the matrix expression  $\sigma_k = (\sigma_k^{qp})_{p,q}, \tau_k = (\tau_k^{qp})_{p,q}$ , where

$$\begin{aligned} \sigma_k^{qp} : V_k^p &\longrightarrow \bigoplus_{h \in \Omega: \text{in}(h)=k} V_{\text{out}(h)}^{q-1} \oplus \bigoplus_{h \in \bar{\Omega}: \text{in}(h)=k} V_{\text{out}(h)}^q \oplus W_k^{q-1}, \\ \tau_k^{qp} : \bigoplus_{h \in \Omega: \text{in}(h)=k} V_{\text{out}(h)}^{p-1} \oplus \bigoplus_{h \in \bar{\Omega}: \text{in}(h)=k} V_{\text{out}(h)}^p \oplus W_k^{p-1} &\longrightarrow V_k^{q-1}. \end{aligned}$$

Components  $\sigma_k^{qp}, \tau_k^{qp}$  vanish if  $q > p$ . From the equation  $\tau_k \sigma_k = 0$ , we have  $\sum_{r: p \leq r \leq q} \tau_k^{qr} \sigma_k^{rp} = 0$ . In particular, we have the complex

$$C_k^{p\bullet} : V_k^p \xrightarrow{\sigma_k^{pp}} \bigoplus_{h \in \Omega: \text{in}(h)=k} V_{\text{out}(h)}^{p-1} \oplus \bigoplus_{h \in \bar{\Omega}: \text{in}(h)=k} V_{\text{out}(h)}^p \oplus W_k^{p-1} \xrightarrow{\tau_k^{pp}} V_k^{p-1}.$$

**Lemma 8.4.** *Let*

$$\bar{\varepsilon}_k^p \stackrel{\text{def.}}{=} - \sum_{q: q > p} \text{rank } C_k^{q\bullet}, \quad \bar{\varphi}_k^p \stackrel{\text{def.}}{=} \sum_{q: q \leq p} \text{rank } C_k^{q\bullet}.$$

*We have*

$$\varepsilon_k(\tilde{\mathfrak{Z}}^\diamond(\rho)) = \max_{p \in \mathbb{Z}} \bar{\varepsilon}_k^p, \quad \varphi_k(\tilde{\mathfrak{Z}}^\diamond(\rho)) = \max_{p \in \mathbb{Z}} \bar{\varphi}_k^p.$$

*Let  $\rho'$  (resp.  $\rho''$ ) is the one parameter subgroup, obtained from  $\rho$ , with  $\dim V_k^p$  decreased (resp. increased) by 1, and other components unchanged, where*

$$\begin{aligned} p &= \min \left\{ q \mid \bar{\varepsilon}_k^q = \varepsilon_k(\tilde{\mathfrak{Z}}^\diamond(\rho)) \right\} \\ &\left( \text{resp. } p = \max \left\{ q \mid \bar{\varphi}_k^q = \varphi_k(\tilde{\mathfrak{Z}}^\diamond(\rho)) \right\} \right). \end{aligned}$$

*Then*

$$\tilde{e}_k(\tilde{\mathfrak{Z}}^\diamond(\rho)) = \begin{cases} 0 & \text{if } \varepsilon_k(\tilde{\mathfrak{Z}}^\diamond(\rho)) = 0, \\ \tilde{\mathfrak{Z}}^\diamond(\rho') & \text{otherwise,} \end{cases} \quad \tilde{f}_k(\tilde{\mathfrak{Z}}^\diamond(\rho)) = \begin{cases} 0 & \text{if } \varphi_k(\tilde{\mathfrak{Z}}^\diamond(\rho)) = 0, \\ \tilde{\mathfrak{Z}}^\diamond(\rho'') & \text{otherwise.} \end{cases}$$

*Proof.* The situation is almost the same as that studied in §4. So  $\varepsilon_k(\tilde{\mathfrak{Z}}^\diamond(\rho))$  is given by the same formula as in the tensor product crystal, if we know the codimension of  $\text{Im } \tau_k^{pp}$  in  $V_k^{p-1}$ . In fact, it was given in [32, 5.5.5]. We have

$$\dim V_k^{p-1} / \text{Im } \tau_k^{pp} = \max(0, -\text{rank } C_k^{p\bullet}) \quad \text{on a generic point } [B, i, j] \in \tilde{\mathfrak{Z}}^\diamond(\rho).$$

(Although the  $\mathbb{C}^*$ -action in [32] is different from our  $\mathbb{C}^*$ -action, the argument works.)

Now we repeat the argument in §4, where  $\text{wt}_k(b_p)$  (resp.  $\varepsilon_k(b_p)$ ) is replaced by  $\text{rank } C_k^{p\bullet}$  (resp.  $\max(0, -\text{rank } C_k^{p\bullet})$ ), and the order of the tensor product is reversed. Therefore, we get

$$\begin{aligned} \varepsilon_k(\tilde{\mathfrak{Z}}^\diamond(\rho)) &= \max_{p \in \mathbb{Z}} \left( \max(0, -\text{rank } C_k^{p\bullet}) - \sum_{q: q > p} \text{rank } C_k^{q\bullet} \right) \\ &= \max_{p \in \mathbb{Z}} \left( - \sum_{q: q \geq p} \text{rank } C_k^{q\bullet} \right) = \max_{p \in \mathbb{Z}} \bar{\varepsilon}_k^p, \end{aligned}$$

and other formulas. □

Let  $\mathbf{w}^p = \sum_k \dim W_k^p \Lambda_k$ ,  $\mathbf{v}^p = \sum_k \dim V_k^p \alpha_k$ . We define a crystal  $\tilde{T}_p$  by

$$\tilde{C}_p \stackrel{\text{def.}}{=} T_{\mathbf{w}^p} \otimes \bigotimes_{k \in I} \mathcal{B}_k,$$

where  $\mathcal{B}_k$  is the crystal in Example 1.6(1), and we have used the numbering of  $I$  to determine the order of the tensor product. Let  $S_0$  be the crystal consisting of a single element  $s_0$  with  $\text{wt } s_0 = 0$ ,  $\varepsilon_k(s_0) = \varphi_k(s_0) = 0$ ,  $\tilde{e}_k s_0 = \tilde{f}_k s_0 = 0$ . We define

$$\tilde{C} \stackrel{\text{def.}}{=} \cdots \otimes \tilde{C}_{p+1} \otimes \tilde{C}_p \otimes \tilde{C}_{p-1} \otimes \cdots.$$

**Theorem 8.5.** *We have a strict embedding of crystal*

$$\begin{aligned} \psi: \text{Irr } \tilde{\mathfrak{Z}}^\diamond &\rightarrow S_0 \otimes \tilde{C} \otimes S_0, \\ \tilde{\mathfrak{Z}}^\diamond(\rho) &\mapsto s_0 \otimes (\cdots \otimes (t_{\mathbf{w}^p} \otimes \otimes_k b_k(-\dim V_k^p)) \otimes \cdots) \otimes s_0. \end{aligned}$$

*Proof.* It is clear that  $\psi$  commutes with  $\text{wt}$ .

Let  $\psi': \text{Irr } \tilde{\mathfrak{Z}}^\diamond \rightarrow \tilde{C}$  be the map defined by omitting the first and last  $s_0$  from the above formula. Let  $b = \cdots \otimes b_{p+1} \otimes b_p \otimes b_{p-1} \otimes \cdots \in \tilde{C}$ . We define  $\varepsilon_k^p, \varphi_k^p$  as in (1.10). If  $b = \psi'(\tilde{\mathfrak{Z}}^\diamond(\rho))$ , then

$$\begin{aligned} \varepsilon_k^p &= \dim V_k^p - \sum_{q \geq p} \dim W_k^q + \sum_{\substack{l \in I: l < k \\ q \geq p}} \langle h_k, \alpha_l \rangle \dim V_l^q \\ &\quad + \sum_{\substack{l \in I: l > k \\ q > p}} \langle h_k, \alpha_l \rangle \dim V_l^q + 2 \sum_{q > p} \dim V_k^q \\ &= - \sum_{q > p} \text{rank } C_k^{q\bullet} = \bar{\varepsilon}_k^p, \end{aligned}$$

where we have used that the property  $h \in \Omega \Rightarrow \text{out}(h) < \text{in}(h)$ ,  $h \in \bar{\Omega} \Rightarrow \text{out}(h) > \text{in}(h)$ .

Thus  $\psi'$  commutes with  $\varepsilon_k$ . From the formula  $\varphi_k^p = \varepsilon_k^p + \text{wt}_k(\cdots \otimes b_p \otimes \cdots)$ , it also commutes with  $\varphi_k$ . Now we get (1.7b, c) for  $\psi'$  by (1.11) and the preceding lemma.

Then we get

$$\begin{aligned} \varepsilon_k(\psi(\tilde{\mathfrak{Z}}^\diamond(\rho))) &= \max(0, \varepsilon_k(\tilde{\mathfrak{Z}}^\diamond(\rho)), -\text{wt}_k(\tilde{\mathfrak{Z}}^\diamond(\rho))) = \varepsilon_k(\tilde{\mathfrak{Z}}^\diamond(\rho)), \\ \varphi_k(\psi(\tilde{\mathfrak{Z}}^\diamond(\rho))) &= \max(0, \varphi_k(\tilde{\mathfrak{Z}}^\diamond(\rho)), \text{wt}_k(\tilde{\mathfrak{Z}}^\diamond(\rho))) = \varphi_k(\tilde{\mathfrak{Z}}^\diamond(\rho)), \end{aligned}$$

where we have used  $\varepsilon_k(\tilde{\mathfrak{Z}}^\diamond(\rho)), \varphi_k(\tilde{\mathfrak{Z}}^\diamond(\rho)) \geq 0$ , which is clear from the definition. We have

$$\tilde{e}_k(\psi(\tilde{\mathfrak{Z}}^\diamond(\rho))) = \begin{cases} s_0 \otimes \psi' \tilde{e}_k(\tilde{\mathfrak{Z}}^\diamond(\rho)) \otimes s_0 & \text{if } \varepsilon_k(\tilde{\mathfrak{Z}}^\diamond(\rho)) \neq 0, \\ 0 & \text{if } \varepsilon_k(\tilde{\mathfrak{Z}}^\diamond(\rho)) = 0. \end{cases}$$

Since  $\varepsilon_k(\tilde{\mathfrak{Z}}^\diamond(\rho)) = 0 \Leftrightarrow \tilde{e}_k(\tilde{\mathfrak{Z}}^\diamond(\rho)) = 0$ ,  $\tilde{e}_k$  commutes with  $\psi$ . Similarly,  $\tilde{f}_k$  commutes with  $\psi$ . □

Let us take  $\rho_0(t) = \text{id}_W$  and hence  $\tilde{\mathfrak{Z}}^\diamond = \mathfrak{L}(\mathbf{w})$ . Then the strictly embedded crystal generated by  $s_0 \otimes \cdots \otimes (t_{\mathbf{w}^p} \otimes \otimes_k b_k(0)) \otimes \cdots \otimes s_0$  in the right hand side is the same as a combinatorial description of the crystal  $\mathcal{B}(\mathbf{w})$  in [14]. Thus we obtain a different proof of Corollary 4.7.

## 9. EXAMPLES

In this section, we give examples of  $\tilde{\mathfrak{Z}}$ .

9.1. Suppose the graph is of type  $A_n$ . We number the vertices as

$$\overset{1}{\bullet} \text{ --- } \overset{2}{\bullet} \text{ --- } \overset{3}{\bullet} \text{ --- } \dots \text{ --- } \overset{n-2}{\bullet} \text{ --- } \overset{n-1}{\bullet} \text{ --- } \overset{n}{\bullet}$$

We take  $\mathbf{w} = r\Lambda_1$ ,  $\mathbf{v} = \sum_{k=1}^n v_k \alpha_k$  with  $r \geq v_1 \geq v_2 \geq \dots \geq v_n \geq 0$ . By [28, 7.3]  $\mathfrak{M}(\mathbf{v}, \mathbf{w})$  is isomorphic to the cotangent bundle  $T^*\mathcal{F}$  of the partial flag variety  $\mathcal{F}$  consisting of all sequences  $\phi = (\mathbb{C}^r = V_0 \supset V_1 \supset \dots \supset V_n \supset V_{n+1} = 0)$  with  $\dim V_k = v_k$ . The correspondence is given by

$$[B, i, j] \mapsto (\phi, \xi) = ((W_1 \supset \text{Im } j_1 \supset \text{Im}(j_1 B_{1,2}) \supset \dots \supset \text{Im}(j_1 B_{1,2} \dots B_{n-1,n}) \supset 0), j_1 i_1),$$

where  $\xi$  is a cotangent vector, i.e., an endomorphism of  $\mathbb{C}^r$  with  $\xi(V_k) \subset V_{k+1}$  ( $k = 0, \dots, n$ ).

Let  $W = W^1 \oplus \dots \oplus W^N$  be a direct sum decomposition. We have the associated flag

$$W = \check{W}^0 \supset \check{W}^1 = \bigoplus_{p>1} W^p \supset \check{W}^2 = \bigoplus_{p>2} W^p \supset \dots \supset \check{W}^{N-1} = W^N \supset \check{W}^N = \{0\}.$$

Then we have

$$\begin{aligned} \tilde{\mathfrak{Z}} &= \{(\phi, \xi) \in T^*\mathcal{F} \mid \xi(\check{W}^p) \subset \check{W}^{p+1}\} \\ \tilde{\mathfrak{Z}}(\mathbf{v}^1, \mathbf{w}^1; \dots; \mathbf{v}^N, \mathbf{w}^N) &= \left\{ (\phi, \xi) \in \tilde{\mathfrak{Z}} \mid \dim(V_k \cap \check{W}^p) = \sum_{q>p} v_k^q \right\}, \end{aligned}$$

where  $v_k^q = \langle h_k, \mathbf{v}^q \rangle$ . Therefore each stratum of  $\tilde{\mathfrak{Z}}$  is the conormal bundle of a Schubert cell.

9.2. Again suppose the graph is of type  $A_n$ . We take  $\mathbf{w} = \Lambda_1 + \Lambda_n$ ,  $\mathbf{v} = \sum_{k=1}^n \alpha_k$ . By a work of Kronheimer,  $\mathfrak{M}(\mathbf{v}, \mathbf{w})$  is the minimal resolution of the simple singularity  $\mathbb{C}^2/\mathbb{Z}_{n+1}$  (see [31, Chapter 4]). The lagrangian subvariety  $\mathfrak{L}(\mathbf{v}, \mathbf{w})$  is the exceptional set. It is a union of  $n$  projective lines. The intersection graph is of type  $A_n$ .

Take coordinates  $(x, y)$  of  $\mathbb{C}^2$  and suppose the action of  $\mathbb{Z}_{n+1}$  is given by  $(x, y) \mapsto (\zeta x, \zeta^{-1}y)$  where  $\zeta$  is a primitive  $(n+1)$ th root of unity. We have a  $\mathbb{C}^*$ -action given by  $(x, y) \mapsto (t^{-1}x, ty)$  commuting with the  $\mathbb{Z}_{n+1}$ -action. This action lifts to an action on  $\mathfrak{M}(\mathbf{v}, \mathbf{w})$  and coincides with the action considered in §3 with  $\mathbf{w}^1 = \Lambda_1$ ,  $\mathbf{w}^2 = \Lambda_n$  (after composed with  $t \mapsto t^{n+1}$ ). Then we have

$$\mathfrak{Z} = \tilde{\mathfrak{Z}} = (\text{the exceptional set}) \cup (\text{the strict transform of the } y\text{-axis}).$$

Each stratum  $\tilde{\mathfrak{Z}}(\mathbf{v}^1, \mathbf{w}^1; \mathbf{v}^2, \mathbf{w}^2)$  is isomorphic to the affine line  $\mathbb{C}$ . The intersection graph of closures of stratum is of type  $A_{n+1}$ , where the strict transform of  $y$ -axis is the last vertex.

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